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Gauge Fields with Torsion[1]

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The Klein–Kaluza theory with a nonvanishing torsion is developed. The torsion is associated with spin and polarization of a gauge field. The electromagnetic polarization is considered as a source of additional components of torsion connected with the fifth dimension. New physical effects obtained due to this torsion are pointed out and some cosmological models are studied. It is proved that new effects are 10^{36} times bigger than the effects from the Einstein–Cartan theory. The usual Dirac equation is generalized to the Klein–Kaluza theory with and without torsion. The dipole electric moment of a fermion of order 10^{-32} cm is obtained. A new generalization of minimal coupling is proposed.

INTRODUCTION

The aim of this paper is to generalize the Klein–Kaluza theory to a situation with nonvanishing torsion of the connection. The polarization of a gauge field and spin will be associated to torsion. Our generalization of the Klein–Kaluza theory is analogous to the relation of the Einstein–Cartan theory to the general theory of relativity. The diagram (Figure 1) places the Klein–Kaluza theory with torsion among the above-mentioned theories.

A new geometric element in our theory is the torsion in the fifth dimension, the source of which is electromagnetic polarization $M_{\alpha\beta}$:

$$Q^{5}{}_{\alpha\beta} = -2K_{\alpha\beta} = 8\pi M_{\alpha\beta}$$

Roughly speaking, if one says that "mass curves space-time," "spin twists it" and "electrical charge curves the fifth dimension" then "electromagnetic

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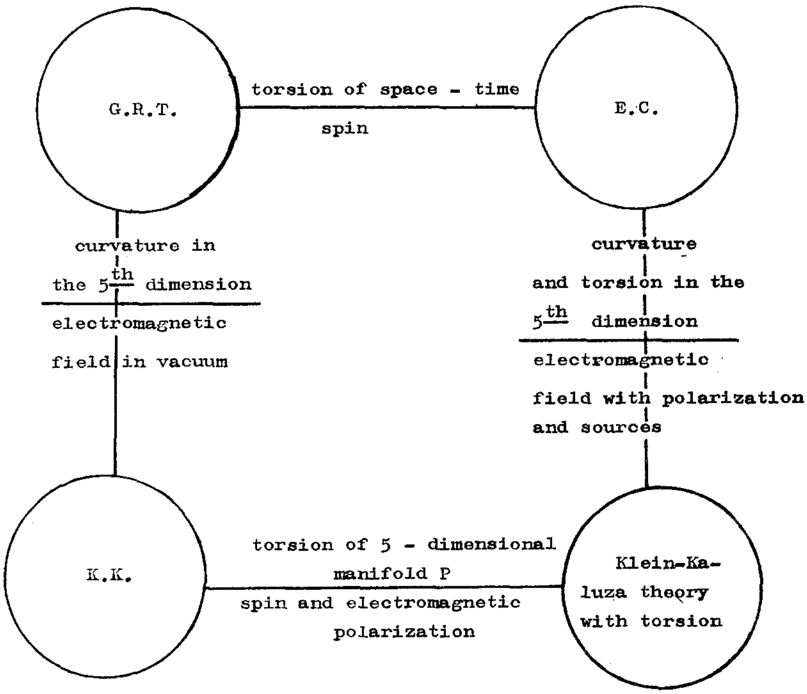

Fig. 1. A position of the Klein–Kaluza theory with torsion among General Relativity, the Einstein–Cartan theory and the classical Klein–Kaluza theory.

polarization twists the fifth dimension." Naturally the fifth dimension is understood as a dimension connected with gauging.

In this paper we will mainly deal with the five-dimensional case (electromagnetic). We shall point out the differences between a general case of gauge fields and electrodynamics. The paper is organized as follows. In Section 1 we define mathematical symbols used throughout the paper. In Section 2 the Klein–Kaluza theory in Cartan's formalism (Lichnerowicz, 1955b) is presented. The main results of this work, i.e., Klein–Kaluza theory with torsion, are given in Section 3. Some physical applications of our theory are given in Section 4 together with a simple cosmological model. In that section we also demonstrate that new effects obtained in the Klein–Kaluza theory with torsion are 10^{36} times bigger than in the Einstein–Cartan theory. We calculate a minimal radius of universe model and get $R \sim 10^{12}$ cm.

Section 5 contains the generalization of Dirac's equation to the Klein–Kaluza theory both with and without torsion. We obtain a dipole electric moment of fermion of order 10^{-32} [cm] q. Our results are similar to Thirring's (1972) work, but we do not have a minimal rest mass of particle as in that work. The procedure proposed in Section 5 may be considered as a generalization of minimal coupling.

In Section 6 we generalize certain results to any semisimple gauge group.

1. ELEMENTS OF GEOMETRY

In this section we describe the notations and definitions of geometric quantities used in the paper. We use a smooth principal bundle P, which includes in its definition the following list of differentiable manifolds and smooth maps: a total (bundle) space P; a Lie group G—structural group [in the electromagnetic case $G=U(1)$]; a base space E–in our case it is a space-time; a projection Π: $P\to E$; a map φ: $P\times G\to P$, defining the action of G on P, if $a, b\in G$ and $\varepsilon\in G$ is the unit element then $\varphi(a)\circ\varphi(b)=\varphi(ba)$ and $\varphi(\varepsilon)=\mathrm{id}$, where $\varphi(a)p=\varphi(p, a)$, moreover $\pi\circ\varphi(a)=\pi$. ω is a form of connection on P with values in Lie's algebra of group G. For a connection of electromagnetic bundle we use a symbol α.

Let $\varphi'(a)$ be the tangent map to $\varphi(a)$ whereas $\varphi^*(a)$ is the contragredient to $\varphi'(a)$ at point a. The form ω is a form of ad type, i.e.,

$$\varphi^*(a)\omega=\mathrm{ad}'_{a^{-1}}\omega \tag{1.1}$$

where $\mathrm{ad}'_{a^{-1}}$ is the tangent map to the internal automorphism of the group

$$\mathrm{ad}_a(b)=aba^{-1}$$

In the case of group $U(1)$ (Abelian) the condition (1.1) means simply

$$\underset{\zeta_5}{\mathfrak{L}_\alpha}=0 \tag{1.2}$$

Where ζ_5 is the Killing vector corresponding to one generator of group $U(1)$. Thus, this is a vector tangent to the operation of group $U(1)$ on P, i.e., to

$$\varphi_{\exp(i\chi)}$$

Because of the form ω we may introduce the distribution (field) of linear elements H_r, $r\in P$, where $H_r\subset T_r(P)$ is a subspace of the space tangent to P at a point r and

$$v\in H_r\Leftrightarrow\omega_r(v)=0 \tag{1.3}$$

We have

$$T_r(P)=V_r\oplus H_r \tag{1.4}$$

where H_r is called a subspace of horizontal vectors and V_r of vertical vectors. For vertical vectors $v\in V_r$ we have $\pi'(V)=0$. This means that v is tangent to the fibers. Let

$$v=\mathrm{hor}(v)+\mathrm{ver}(v), \qquad \mathrm{hor}(v)\in H_r, \qquad \mathrm{ver}(v)\in V_r \tag{1.5}$$

It is proved that the distribution H_r is equal to choosing connection ω. We use the operation "hor" for forms, i.e.,

$$(\mathrm{hor}\,\beta)(X,Y)=\beta(\mathrm{hor}\,X,\mathrm{hor}\,Y) \tag{1.6}$$

where

$$H,Y\in T(P)$$

The two-form of curvature of connection ω is defined as follows:

$$\Omega=\mathrm{hor}\,d\omega \tag{1.7}$$

It is also a form of ad type like ω.
For Ω the structural Cartan equation is valid:

$$\Omega=d\omega+\tfrac{1}{2}[\omega,\omega] \tag{1.8}$$

where

$$[\omega,\omega](X,Y)=[\omega(X),\omega(Y)]$$

Bianchi's identity for ω is the following:

$$\mathrm{hor}\,d\Omega=0 \tag{1.9}$$

For the principal fiber bundle we use the following convenient scheme (Figure 2):

$$\text{The map } f\colon E\supset V\to P, \qquad \text{so that } f\circ\pi=\mathrm{id}$$

is called a cross section. From the physical point of view it means choosing a gauge. A covariant derivation on P d_1 is defined as follows:

$$d_1\Psi=\mathrm{hor}\,d\Psi \tag{1.10}$$

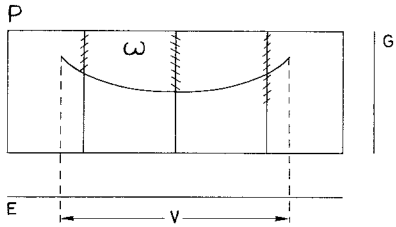

Fig. 2. Vertical lines depict fibers, while cross-hatched lines indicate the distribution of linear elements H_r.

This derivation is called a "gauge" derivation, where Ψ is for example a spinor field on P.

For a principal fiber bundle P it is possible to introduce a natural metrization in the following way:

$$\gamma(X,Y)=g(\pi'X,\pi'Y)-\lambda^2 h(\omega(X),\omega(Y)) \qquad (1.11)$$
$$X,Y\in T(P), \qquad 0<\lambda=\text{const}$$

Where h is the Killing tensor on a group G. It is obvious that G must be semisimple. We have $h_{ab}=C^c_{ad}C^d_{bc}$, where C^c_{ab} are structural constants of Lie's algebra of group G. The formula (1.11) has been given by A. Trautman (1970). In the case of group $U(1)$ we have number one -1 as biinvariant tensor h. The tensor γ is invariant with respect to group G. In this paper we will use also a linear connection on manifolds P and E using formalisms of differential forms. So the basic quantity is a one-form of connection ω_{AB}. The two-form of curvature is as follows:

$$\Omega^A{}_B=d\omega^A{}_B+\omega^A{}_C\wedge\omega^C{}_B \qquad (1.12)$$

and the two-form of torsion is

$$\Theta^A=D\theta^A \qquad (1.13)$$

where θ^A are basic forms and where D means exterior covariant derivation with respect to ω_{AB}. The following relation are established connections with generally met symbols:

$$\omega^A{}_B=\Gamma^A{}_{BC}\theta^C$$
$$\Theta^A=\tfrac{1}{2}Q^A{}_{BC}\theta^B\wedge\theta^C$$
$$Q^A{}_{BC}=\Gamma^A{}_{BC}-\Gamma^A{}_{CB}$$
$$\Omega^A{}_B=\tfrac{1}{2}R^A{}_{BCD}\theta^C\wedge\theta^D \qquad (1.14)$$

where $\Gamma^A{}_{BC}$ are coefficients of connection (they do not have to be symmetrical indices B and C), $R^A{}_{BCD}$ is a tensor of curvature, and $Q^A{}_{BC}$ is a tensor of torsion. Covariant exterior derivation with respect to $\omega^A{}_B$ is given by a formula:

$$D\Xi^A = d\Xi^A + \omega^A{}_C \wedge \Xi^C$$

$$D\Sigma^A{}_B = d\Sigma^A{}_B + \omega^A{}_C \wedge \Sigma^C{}_B - \omega^C{}_B \wedge \Sigma^A{}_C \tag{1.15}$$

The forms of curvature Ω^A_B and torsion Θ^A obey Bianchi's identities

$$D\Omega^A_B = 0$$

$$D\Theta^A = \Omega^A{}_B \wedge \theta^B \tag{1.16}$$

All quantities introduced in this paper and their precise definitions can be found in the papers by Kobayashi and Nomizu (1963), Lichnerowicz (1955b), and Trautman (1970, 1973b).

2. THE KLEIN–KALUZA THEORY

2.1. Preliminary Remarks. In this chapter we present the classical Klein–Kaluza theory by means of the mathematical methods mentioned in Section 1. In the papers by Kaluza (1921), Lichnerowicz (1955a), Rayski (1965), Tonnelat (1965), and Bergman (1942) one may find both consecutive steps of creation of the theory and various approaches to it. The final form of the theory was achieved in the paper by Bergman (1942). Its particular variant, the so-called Jordan–Thiry theory, was proposed in the paper by Lichnerowicz (1955a). The Jordan–Thiry theory has much to do with Brans–Dick scalar–tensor theory of gravitation (Bergman, 1968) because of the introduction of an additional scalar field h (interpretation of h^2 is a gravitational "constant"). The equivalence of the Klein–Kaluza and Utiyama theories (Lichnerowicz, 1955b) of gauge fields has been discovered by Trautman and Tulczyjew (1970). Naturally Utiyama's approach is more general and makes possible a creation of unified theories of Yang–Mills field and gravitation. It has been done in Kerner (1968) and Cho (1975). It would be possible to achieve these results by means of conventional methods similar to methods given in Lichnerowicz (1955a), Bergman (1942), and Tonnelat (1965). In order to do it one should take into account a $(n+4)$-dimensional manifold (n is the number of parameters of gauge group of the Yang–Mills field). In this section we use the differential form method in a way similar to that proposed already by A. Trautman (1972, 1973a). Section 2 serves for comparison to the results of Section 3.

2.2. Manifold *P*. Let us introduce the principal fiber bundle P over space-time E with the group structure $U(1)$ and with projection π. It is an electromagnetic bundle (see Figure 3).

Using Cartan's structural equation (1.8) and Abelian character of $U(1)$ we obtain curvature of connection:

$$\Omega = d\alpha \tag{2.1}$$

Now let us take two sections:

$$e: E \to P, \qquad f: E \to P$$

In both cases we have

$$\begin{aligned} A &= e^*\alpha, \qquad F = e^*\Omega \\ \bar{A} &= f^*\alpha, \qquad \bar{F} = f^*\Omega \end{aligned} \tag{2.2}$$

Since we may identify Lie's algebra of group $U(1)$ with real numbers, forms $F, A, \bar{A}, \bar{F}$, are ordinary forms with real values. The form F due to the Abelian character of group $U(1)$ does not depend on choosing a section and $F = \bar{F}$. We have indeed

$$F = dA, \qquad \bar{F} = d\bar{A} \tag{2.3}$$

Let χ: $E \to R$ be a change of section from e to f.

$$f(p) = \varphi_{\exp[i\chi(p)]} \circ e(p) \tag{2.4}$$

Thus we have

$$\bar{A} = A + d\chi \tag{2.5}$$

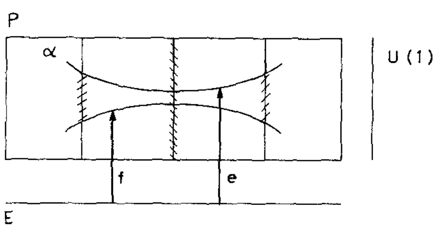

Fig. 3. α is a connection on bundle P.

Form A is a form of four-potential

$$A = A_\mu \bar{\theta}^\mu \tag{2.6}$$

whereas

$$F = \tfrac{1}{2} F_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.7}$$

is a 2-form of strength of electromagnetic field, where $\bar{\theta}^\mu$ is a frame on E. We have $dF = 0$ that is equivalent to Bianchi's identity for the connection α. This equation is equivalent to the first pair of Maxwell equations, i.e., to the condition of 4-potential existence. The connection α is called an electromagnetic connection.

Simultaneously, we also have

$$\Omega = \pi^*(F) \tag{2.8}$$

Now we turn to metrization of bundle P. Let us suppose that (E, g) is a manifold with a metric tensor g, and Riemann connection $\tilde{\bar{\omega}}\alpha\beta$, where $g = g_{\alpha\beta} \bar{\theta}^\alpha \otimes \bar{\theta}^\beta$. The signature of g is $(- - - +)$ and Θ^α is a frame on E.

Let us introduce a frame on P:

$$\theta^A = \big(\pi^*(\bar{\theta}^\alpha), \theta^5 = \lambda\alpha\big), \qquad \lambda > 0, \text{const} \tag{2.9}$$

It is convenient to introduce the following notations: Capital Latin indices $A, B, C, D, E = 1, 2, 3, 4, 5$. Lower case Greek indices $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$. The symbol $\simeq$ is introduced to indicate two properties of $\tilde{\bar{\omega}}_{\alpha\beta}$: the Riemannian feature and the fact that it is defined on E. Let us now introduce the tensor $\gamma = \gamma_{AB} \theta^A \otimes \theta^B$ on the manifold P in the natural way (Trautman, 1970). Let $X, Y \in \mathrm{Tan}(P)$. Thus according to the formula (1.11) we have

$$\gamma(X, Y) = g(\pi' X, \pi' Y) - \lambda^2 \alpha(X) \alpha(Y) = g(\pi' X, \pi' Y) - \theta^5(X) \theta^5(Y)$$

or

$$\gamma = \pi^* g - \theta^5 \otimes \theta^5 \tag{2.10}$$

Tensor γ has signature $(- - - + -)$. In this particular frame this tensor has a form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & -1 \end{array} \right) \tag{2.11}$$

It is clear that the frame θ^A is partly unholonomical, because

$$d\theta^5 = \pi^*(\lambda F) \neq 0 \tag{2.12}$$

We also introduce a dual frame (ζ_A):

$$\gamma(\zeta_A) = \gamma_{AB}\theta^B \tag{2.13}$$

We have $\zeta_A = (\zeta_\alpha, \zeta_5)$ and according to Section 1

$$\begin{aligned} &\alpha\gamma = 0 \\ &\zeta_5 \end{aligned} \tag{2.14}$$

Thus ζ_5 is Killing's vector of metric γ. Let us now introduce the Riemann connection $\tilde{\omega}_{AB}$ on P and exterior covariant derivation D with respect to $\tilde{\omega}_{AB}$:

$$\tilde{D}\gamma_{AB} = 0, \qquad \tilde{D}\theta^A = 0 \tag{2.15}$$

A solution of (2.15) is

$$\tilde{\omega}_{AB} = \left(\begin{array}{c|c} \pi^*(\tilde{\bar{\omega}}_{\alpha\beta}) + \frac{1}{2}\lambda\pi^*(F_{\alpha\beta})\theta^5 & \frac{1}{2}\lambda\pi^*(F_{\alpha\gamma}\theta^\gamma) \\ \hline -\frac{1}{2}\lambda\pi^*(F_{\beta\gamma}\theta^\gamma) & 0 \end{array}\right) \tag{2.16}$$

Now we define a dual Cartan base on P. Let $\eta_{12345} = (\det\gamma)^{1/2} = (-\det g)^{1/2}$ and η_{ABCDE} be a Levi-Cività symbol and

$$\begin{aligned} \eta_{ABCD} &= \theta^E \eta_{ABCDE} \\ \eta_{ABC} &= \tfrac{1}{2}\theta^D \wedge \eta_{ABCD} \\ \eta_{AB} &= \tfrac{1}{3}\theta^C \wedge \eta_{ABC} \\ \eta_A &= \tfrac{1}{4}\theta^B \wedge \eta_{AB} \\ \eta &= \tfrac{1}{5}\theta^A \wedge \eta_A \end{aligned} \tag{2.17}$$

On a manifold (E, g), i.e., on a space-time we introduce analogous quantities:

$$\bar{\eta}_{\alpha\beta\gamma\delta}, \qquad \bar{\eta}_{1234} = (-\det g)^{1/2} \tag{2.18}$$

$$\bar{\eta}_{\alpha\beta\gamma}=\theta^{\delta}\bar{\eta}_{\alpha\beta\gamma\delta}$$

$$\bar{\eta}_{\alpha\beta}=\tfrac{1}{2}\bar{\theta}^{\gamma}\wedge\bar{\eta}_{\alpha\beta\gamma}$$

$$\bar{\eta}_{\alpha}=\tfrac{1}{3}\bar{\theta}^{\beta}\wedge\bar{\eta}_{\alpha\beta}$$

$$\bar{\eta}=\tfrac{1}{4}\bar{\theta}^{\alpha}\wedge\bar{\eta}_{\alpha} \tag{2.19}$$

Now we can define quantities which appeared in (2.17) by (2.19) and Θ^5.

$$\eta_{\alpha\beta\gamma\delta\varepsilon}=0$$

$$\eta_{\alpha\beta\gamma\delta5}=\bar{\eta}_{\alpha\beta\gamma\delta}$$

$$\eta_{\alpha\beta\gamma\delta}=\bar{\eta}_{\alpha\beta\gamma\delta}\wedge\theta^{5}$$

$$\eta_{\alpha\beta5}=-\bar{\eta}_{\alpha\beta\gamma}$$

$$\eta_{\alpha\beta\gamma}=\bar{\eta}_{\alpha\beta\gamma}\wedge\theta^{5}$$

$$\eta_{\alpha\beta5}=\bar{\eta}_{\alpha\beta}$$

$$\eta_{\alpha\beta}=\bar{\eta}_{\alpha\beta}\wedge\theta^{5}$$

$$\eta_{\alpha5}=-\bar{\eta}_{\alpha}$$

$$\eta_{\alpha}=\bar{\eta}_{\alpha}\wedge\theta^{5}$$

$$\eta_{5}=\bar{\eta}$$

$$\eta=\bar{\eta}\wedge\theta^{5} \tag{2.20}$$

In the formulas (2.20) on the left-hand side there are quantities defined on P and on the right-hand side defined on E. Actually it should be denoted not $\bar{\eta}$ but $\pi^*(\bar{\eta})$, etc. For brevity we shall omit horizontal lift π^* in all cases where it will not lead to a misunderstanding. In the case (2.20) we should remember that the overbar placed above a geometrical quantity means that it is defined on E. Quantities $\bar{\eta}_{\alpha\beta\gamma}$, $\bar{\eta}_{\alpha\beta}$, etc. obey the following identities:

$$\bar{\theta}^{\varepsilon}\bar{\eta}_{\alpha\beta\gamma\delta}=\delta^{\varepsilon}_{\delta}\bar{\eta}_{\alpha\beta\gamma}-\delta^{\varepsilon}_{\gamma}\bar{\eta}_{\delta\alpha\beta}+\delta^{\varepsilon}_{\beta}\bar{\eta}_{\gamma\delta\varepsilon}-\delta^{\varepsilon}_{\alpha}\bar{\eta}_{\beta\gamma\delta}$$

$$\bar{\theta}^{\varepsilon}\wedge\bar{\eta}_{\alpha\beta\gamma}=\delta^{\varepsilon}_{\gamma}\bar{\eta}_{\alpha\beta}+\delta^{\varepsilon}_{\beta}\bar{\eta}_{\gamma\alpha}+\delta^{\varepsilon}_{\alpha}\bar{\eta}_{\beta\gamma}$$

$$\bar{\theta}^{\varepsilon}\wedge\bar{\eta}_{\alpha\beta}=\delta^{\varepsilon}_{\beta}\bar{\eta}_{\alpha}-\delta^{\varepsilon}_{\alpha}\bar{\eta}_{\beta} \tag{2.21}$$

Now let us take a section e: $E \to P$ and fit to it a coordinate x^5, selecting $x^\mu = \text{const}$ on the fiber in such a way that e is given by the condition $x^5 = 0$, and

$$\zeta_5 = \frac{\partial}{\partial x^5}$$

(see Figure 4). Then we have $e^* dx^5 = 0$ and

$$\alpha = \frac{1}{\lambda} dx^5 + \pi^*\left(A_\mu \bar{\theta}^\mu\right), \qquad \text{where } A = A_\mu \bar{\theta}^\mu = e^*\alpha$$

In this coordinate system tensor γ takes the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} - \lambda^2 A_\alpha A_\beta & -\lambda A_\alpha \\ \hline -\lambda A_\beta & -1 \end{array}\right) \tag{2.22}$$

This coordinate system is holonomic.

Roughly speaking classically defined Klein–Kaluza theory consists in building a five-dimensional analog of the general theory of relativity, with metric tensor in the form (2.22).

Now we calculate a two-form of curvature $\tilde{\Omega}^A{}_B$ of $\tilde{\omega}^A{}_B$. By applying formulas (1.12) and (2.16) we obtain

$$\begin{aligned} &\tilde{\Omega}^\alpha{}_\beta = \bar{\tilde{\Omega}}^\alpha{}_\beta + \tfrac{1}{2}\lambda \bar{\tilde{D}} F^\alpha{}_\beta \wedge \theta^5 + \tfrac{1}{4}\lambda\left(F^\alpha{}_\beta F_{\mu\nu} + F^\alpha{}_\mu F_{\beta\nu}\right)\bar{\theta}^\mu \wedge \bar{\theta}^\nu \\ &\tilde{\Omega}^5{}_\beta = \tfrac{1}{2}\lambda \bar{\tilde{D}} F_\beta + \tfrac{1}{4}\lambda^2\left(F_{\gamma\mu} F_\beta^\gamma \bar{\theta}^\mu\right) \wedge \theta^5 = -\tilde{\Omega}^5{}_\beta \\ &\tilde{\Omega}^5{}_5 = 0 \end{aligned} \tag{2.23}$$

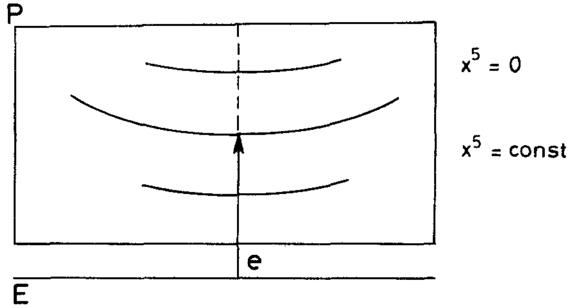

Fig. 4. A connection between the fiber bundle formulation of the Klein–Kaluza theory and the classical formulation.

where $\tilde{\tilde{\Omega}}^{\alpha}{}_{\beta}$ is a two-form of curvature of $\tilde{\tilde{\omega}}^{\alpha}{}_{\beta}$, $\tilde{\tilde{D}}$ is an exterior covariant differential with respect to $\tilde{\tilde{\omega}}^{\alpha}{}_{\beta}$, and $F_{\beta}=F_{\beta\alpha}\Theta^{\alpha}$.

Finally we give below coefficients of $\tilde{\tilde{\omega}}_{AB}$:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma}=\tilde{\tilde{\Gamma}}^{\alpha}_{\beta\gamma}=\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$$

$$\tilde{\Gamma}^{5}_{\beta\gamma}=\tfrac{1}{2}\lambda F_{\beta\gamma}$$

$$\tilde{\Gamma}^{\alpha}_{\beta 5}=\tilde{\Gamma}^{\alpha}_{5\beta}=\tfrac{1}{2}\lambda F^{\alpha}{}_{\beta} \tag{2.24}$$

where $\tilde{\tilde{\Gamma}}^{\alpha}_{\beta\gamma}$ are coefficients of connection $\tilde{\tilde{\omega}}_{\alpha\beta}$. The rest of the coefficients are equal to zero.

2.3. Variational Principle and Field Equations. Now we shall derive equations of the Klein–Kaluza theory from the variational principle for scalar of curvature $\tilde{K}$ with respect to $\theta^{A}, \gamma_{AB}, \tilde{\omega}^{A}{}_{B}$. We take

$$\tilde{K}=\tfrac{1}{2}\eta_{AB}\wedge\tilde{\Omega}^{AB} \tag{2.25}$$

It follows from the relationship between γ and Θ^{5}, that only $\Theta^{\alpha}, \Theta^{5}$, $g_{\alpha\beta}$ are independent quantities. Thus we do not vary $\tilde{K}$ with respect to $\gamma_{\alpha 5}$ and γ_{55}. We have

$$\delta\tilde{K}=\delta\theta^{A}\wedge\tilde{e}_{A}+\tfrac{1}{2}\delta g_{\alpha\beta}\tilde{E}^{\alpha\beta}-\tfrac{1}{2}\delta\tilde{\omega}^{A}{}_{B}\wedge p_{A}{}^{B}+\text{exact form} \tag{2.26}$$

In the formula (2.26) $\tilde{e}_{A}$, $\tilde{E}^{\alpha\beta}$, $\tilde{p}_{A}{}^{B}$ are five-dimensional analogs of Einstein's form $\tilde{\tilde{e}}_{\alpha}$, symmetrical Einstein's form $\tilde{\tilde{E}}^{\alpha\beta}$, and Palatini's form $\bar{p}^{\alpha\beta}$, respectively. The latter are known from the Einstein–Cartan theory (Trautman, 1972, 1973a; Kopczyński, 1973). Performing variation $\bar{K}$ with respect to $\Theta^{A}, g^{\alpha\beta}, \tilde{\omega}^{A}{}_{B}$ we obtain

$$\tilde{e}_{A}=-\tfrac{1}{2}\eta^{B}{}_{CA}\wedge\tilde{\Omega}^{C}{}_{B} \tag{2.27}$$

$$\tilde{E}^{\alpha\beta}=\tfrac{1}{2}\left(\gamma^{\alpha\beta}\eta_{C}{}^{D}-\gamma^{\alpha D}\eta_{C}{}^{\beta}-\gamma^{\beta D}\eta_{C}{}^{A}\right)\wedge\tilde{\Omega}^{C}{}_{D} \tag{2.28}$$

$$\tilde{p}_{A}{}^{B}=-\tilde{D}\eta^{B}{}_{A} \tag{2.29}$$

By using the formulas (2.11), (2.16), (2.20), (2.21), (2.23) we obtain the following equations:

$$\tilde{e}_\alpha = \tilde{\tilde{e}}_\alpha \wedge \theta^5 + \left(\tfrac{1}{8}\lambda^2 F^{\mu\nu} F_{\mu\nu} \bar{\eta}_\alpha + \tfrac{1}{2}\lambda^2 F^{\gamma\mu} F_{\mu\alpha} \bar{\eta}_\gamma\right) \wedge \theta^5 + \tfrac{1}{2}\lambda \bar{\tilde{\nabla}}_\gamma F^{\gamma}_{\alpha} \bar{\eta} \quad (2.30)$$

$$\tilde{e}_5 = \tfrac{1}{2}\lambda \bar{\tilde{\nabla}}_\beta F^{\beta\gamma} \bar{\eta}_\gamma \wedge \theta^5 + \tfrac{1}{2}\bar{\eta}_{\gamma\beta} \wedge \tilde{\tilde{\Omega}}^{\gamma\beta} + \tfrac{3}{8}\lambda^2 F_{\mu\nu} F^{\mu\nu} \bar{\eta} \quad (2.31)$$

$$\tilde{E}^{\alpha\beta} = \tilde{\tilde{E}}^{\alpha\beta} \wedge \theta^5 + \left(-\tfrac{1}{2}\lambda F^{\mu\alpha} F^{\beta}_{\mu} + \tfrac{1}{8}\lambda^2 F^{\mu\nu} F_{\mu\nu} g^{\alpha\beta}\right) \bar{\eta} \wedge \theta^5 \quad (2.32)$$

$$p_\alpha{}^\beta = p^\alpha{}_5 = p^5{}_\beta = p^5{}_5 = 0 \quad (2.33)$$

The last equations (2.33) are valid because of the Riemannian property of $\tilde{\omega}_{AB}$, where $\tilde{\tilde{E}}^{\alpha\beta}$ and $\tilde{\tilde{e}}^\alpha$ are, respectively, Einstein's symmetrical form and Einstein's 3-form given by formulas (Trautman, 1970)

$$\tilde{\tilde{E}}^{\alpha\beta} = \tfrac{1}{2}\left(g^{\alpha\beta}\bar{\eta}_\gamma^{\ \delta} - g^{\alpha\delta}\bar{\eta}_\gamma^{\ \beta} - g^{\beta\delta}\bar{\eta}_\gamma^{\ \alpha}\right) \wedge \tilde{\tilde{\Omega}}^\gamma{}_\delta \quad (2.34)$$

$$\tilde{\tilde{e}}_\alpha = -\tfrac{1}{2}\bar{\eta}^\beta{}_{\gamma\alpha} \wedge \tilde{\tilde{\Omega}}^\gamma{}_\beta \quad (2.35)$$

Now let us turn to field equations. They are introduced into the Klein–Kaluza theory as sourceless equations:

$$\tilde{E}^{\alpha\beta} = 0 \qquad \text{and} \qquad \tilde{e}^A = 0 \quad (2.36)$$

Hence, taking into account (2.30), (2.31), (2.32), we obtain

$$\tilde{\tilde{E}}^{\alpha\beta} = \lambda^2\left(\tfrac{1}{2}F^{\mu\alpha}F^{\beta}_{\mu} - \tfrac{1}{8}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right)\bar{\eta} \quad (2.37)$$

$$\tilde{\tilde{e}}_\alpha = \lambda^2\left(-\tfrac{1}{2}F^{\gamma\mu}F_{\mu\alpha}\bar{\eta}_\gamma - \tfrac{1}{8}F^{\mu\nu}F_{\mu\nu}\bar{\eta}_\alpha\right) \quad (2.38)$$

$$\bar{\tilde{\nabla}}_\beta F^{\beta\gamma} = 0 \quad (2.39)$$

From the equation $\tilde{e}^5 = 0$ we get one more equation which has not been given above. This equation does not belong to the equations of the Klein–Kaluza theory and is related to the equation $G_{55} = 0$, where G_{AB} is the Einstein tensor. This equation should be deleted by taking $\tilde{e}^5 - \text{hor}\,\tilde{e}^5$ instead of $\tilde{e}^5$. The fact that the equation $G_{55} = 0$ appears is related to constraints existing in the theory, i.e., $\gamma_{55} = -1$ and $\gamma_{\alpha 5} = 0$. An appropriate theory of Lagrange's multipliers might remove this drawback. However, we shall not deal with it in detail in this paper. The equality $\gamma_{55} = -1$ is related to the assumption that λ is constant. In a theory in which λ may be variable, e.g., in the Jordan–Thiry theory, one obtains by varying $\tilde{K}$ with respect to γ_{55} an equation for λ. This leads to a theory with gravitational "constant" variable in time and space. It is just a theory of the Brans–Dick type. Now consider the equations (2.37) and (2.38).

Notice that

$$\tfrac{1}{2}F^{\mu\alpha}F^{\beta}_{\mu}-\tfrac{1}{8}F^{\mu\nu}F_{\mu\nu}g_{\alpha\beta}=2\pi \overset{\text{em. vac.}}{T^{\alpha\beta}} \tag{2.40}$$

and

$$2\pi \overset{\text{em. vac.}}{t_{\alpha}} = -\tfrac{1}{2}F^{\gamma\mu}F_{\mu\alpha}\bar{\eta}_{\gamma}-\tfrac{1}{8}F^{\mu\nu}F_{\mu\nu}\bar{\eta}_{\alpha} \tag{2.41}$$

where $\overset{\text{em. vac.}}{T^{\alpha\beta}}$ is a tensor of energy-momentum of electromagnetical field in vacuum and $\overset{\text{em. vac.}}{t_{\alpha}}$ is a form of energy-momentum of electromagnetical field in vacuum.

We easily see that

$$\overset{\text{em. vac.}}{T^{\alpha\beta}} = \overset{\text{em. vac.}}{t^{\alpha\beta}}, \quad \text{where} \quad \overset{\text{em. vac.}}{t^{\alpha}} = \overset{\text{em. vac.}}{t^{\alpha\beta}}\bar{\eta}_{\beta} \tag{2.42}$$

Putting $\lambda=2$ we obtain Einstein's equations with tensor of energy-momentum of electromagnetic field in vacuum as a source. Quantities in these equations are in units of the theoretical Gauss system with $G=1$ and $c=1$

$$\bar{\tilde{E}}^{\alpha\beta}=8\pi \overset{\text{em. vac.}}{T^{\alpha\beta}}\ \bar{\eta}$$

$$\bar{\tilde{e}}^{\alpha}=8\pi \overset{\text{em. vac.}}{t^{\alpha}} \tag{2.43}$$

The last equations are equivalent to one of Einstein's equations:

$$\bar{\tilde{R}}^{\alpha\beta}-\tfrac{1}{2}\bar{\tilde{R}}g_{\alpha\beta}=8\pi \overset{\text{em. vac.}}{T^{\alpha\beta}} \tag{2.44}$$

In order to go back to units of the cgs system we should take $\lambda=2G^{1/2}/c^{2}$. We shall use the theoretical system of units also in Section 3.

The equation (2.39) gives us the second pair of Maxwell equations. The first pair of Maxwell equations has been obtained earlier. Summing up, we have achieved equations of the Klein–Kaluza theory, i.e., equations of gravitation and electromagnetism.

2.4. Bianchi's Identities. In the case of Riemannian geometry only the first of equations (1.16) is interesting for us. From this result the so-called contracted Bianchi's identities $\tilde{D}e_A=0$, and they lead to the conservation law of energy-momentum of electromagnetic field:

$$\tilde{\bar{D}}\overset{\text{em}}{t^{\alpha}}=0 \tag{2.45}$$

and to the identity

$$\tilde{\bar{\nabla}}^{\alpha}\left(\tilde{\bar{\nabla}}_{\gamma}F_{\alpha}^{\gamma}\right)=0 \tag{2.46}$$

The second identity is not interesting for a lack of sources of electromagnetic field.

2.5. Geodetic Lines. Finally we discuss the equations of geodetic lines on manifold P

$$u^{B}\tilde{\nabla}_{B}u^{A}=0 \tag{2.47}$$

where $u^A(t)$ is a vector normalized as follows:

$$g_{\alpha\beta}u^{\alpha}u^{\beta}=1 \quad \text{and} \quad g_{AB}u^{A}u^{B}=1-(u^{5})^{2} \tag{2.48}$$

A trajectory tangent to this vector field is a geodetic line. The normalization condition (2.48) is easily understood if we take into account the existence of Killing's vector ζ_5. This leads to an existence of the first integral of equation (2.47) $u^5=\text{const}$.

Putting (2.24) to (2.47) and using $\lambda=2$ we have

$$\frac{\tilde{\bar{D}}u^{\alpha}}{dt}+2u^{5}F_{\beta}^{\alpha}u^{\beta}=0, \qquad u^{5}=\text{const} \tag{2.49}$$

where D/dt is a covariant derivation along a line to which u^α is tangent. The first equation of (2.51) is an equation of motion of a matter point of $q/m_0=2u^5$ in both gravitational and electromagnetic fields (q is electric charge and m_0 is a rest mass). The second equation of (2.49) means constancy of q/m_0 along the world line of a particle.

3. THE KLEIN–KALUZA THEORY WITH TORSION

3.1. Preliminary Remarks. The aim of this section is both the generalization of the Klein–Kaluza theory to the case with a nonvanishing torsion of connection and finding physical interpretation of this torsion. The general plan is the following. We introduce on P the connection with nonvanishing torsion. This connection is invariant with respect to transformations of group $U(1)$. In this paper we also assume that a torsion of the connection is horizontal. The last condition will be discussed. Next we construct a form of a scalar curvature for this connection and introduce sources. Then from a variational principle we obtain equations of fields and interpret them. According to the postulate of geometrization of physical quantities we shall obtain equations where on the left-hand side there will be geometrical quantities and on the right-hand side matter quantities. In this way matter quantities will be sources of geometry. We shall obtain an interpretation of electromagnetic polarization as a torsion related to the fifth dimension. We get equations of gravitation in the Einstein–Cartan theory. On the right-hand side as source will be the sum of energy-momentum tensors of electromagnetic field with polarization of matter in the form given by W. Israel (Bailey and Israel, 1975; Israel, 1977, 1974) and of matter. Additionally there will be also a component $\pi g_{\alpha\beta} M_{\mu\nu} M^{\mu\nu}$, where $M_{\mu\nu}$ is a tensor of electromagnetic polarization of matter. This additional component has been obtained similarly, as the component with contact interaction (spin)×(spin) in the Einstein–Cartan theory. The new component may be treated as a contact interaction (electromagnetic moment)× (electromagnetic moment). The role of this component will be estimated and compared with the effects originated from Einstein–Cartan theory in Section 4. An equation of motion of a charged particle in an electromagnetic field in space with torsion will be derived from the equation of a geodetic line. We also derive the second pair of Maxwell equations in terms of derivatives with respect to connection with torsion. This will give us an additional internal current related to spin. The role of this current will be estimated in Sections 4 and 5. From Bianchi's identity we get conservation laws of energy-momentum, angular momentum, and charge. In this chapter we also analyze an example of an application of this theory, i.e., the vector field on the five-dimensional manifold P.

3.2. Formulation of the Problem. We introduce an electromagnetic bundle P with natural metrization and a metrical connection $\omega^A{}_B$ on P invariant under a transformation of $U(1)$. The connection $\omega^A{}_B$ is not necessarily Riemannian. We also define a connection $\bar{\omega}^{\alpha\beta}$ metrical, but not necessarily Riemannian on E. As far as indices are concerned, we follow the conventions given in Section 2. We assume that an overbar above a symbol denoting connection, covariant derivation, curvature, or some other quantity indicates that the quantity is defined on E, whereas a tilde means a quantity depending on Riemannian connection, e.g., $\tilde{\bar{\omega}}^{\alpha}{}_{\beta}$ means Riemannian connection on E. Now we have $(E, g, \bar{\omega}^{\alpha}{}_{\beta})$ a four-dimensional manifold with metrical connection, metrical tensor g with the signature $(---+)$, and $(P, \gamma, \omega^{A}{}_{B})$ a five-dimensional manifold with the metrical connection $\omega^{A}{}_{B}$. Thus

$$D\gamma_{AB}=0, \qquad \underset{\zeta_5}{\mathcal{L}}\omega^{A}{}_{B}=0 \tag{3.1}$$

Separating $\omega_{AB}=\tilde{\omega}_{AB}+\kappa_{AB}$ into a Riemannian part and a defect κ_{AB} we have by virtue of (3.1)

$$\kappa_{AB}=-\kappa_{BA} \tag{3.2}$$

$$\underset{\zeta_5}{\mathcal{L}}\kappa_{AB}=0 \tag{3.3}$$

The most general form of κ_{AB} obeying the equations (3.2) and (3.3) is

$$\kappa_{AB}=\left(\begin{array}{c|c} \pi^*(\bar{\kappa}_{\alpha\beta})+\pi^*(K_{\alpha\beta})\theta^5 & \pi^*(L_{\alpha\gamma}\bar{\theta}^{\gamma})+\pi^*(Z_{\alpha})\theta^5 \\ \hline -\pi^*(L_{\beta\gamma}\bar{\theta}^{\gamma})-\pi^*(Z_{\beta})\theta^5 & 0 \end{array}\right) \tag{3.4}$$

$L_{\beta\gamma}$, $K_{\alpha\beta}=-K_{\beta\alpha}$, Z_{α} are tensors on E, whereas $\bar{\kappa}_{\alpha\beta}=-\bar{\kappa}_{\beta\alpha}$ is a defect of connection $\bar{\omega}_{\alpha\beta}$ on E and

$$\bar{\omega}_{\alpha\beta}=\tilde{\bar{\omega}}_{\alpha\beta}+\bar{\kappa}_{\alpha\beta} \tag{3.5}$$

The most general form of a connection satisfying the condition (3.1) is

$$\begin{aligned} &\omega_{\alpha\beta}=\pi^*(\bar{\omega}_{\alpha\beta})+\pi^*(F_{\alpha\beta}+K_{\alpha\beta})\theta^5 \\ &\omega_{\alpha5}=-\omega_{5\alpha}=\pi^*\left[(F_{\alpha\gamma}+L_{\alpha\gamma})\bar{\theta}^{\gamma}\right]+\pi^*(Z_{\alpha})\theta^5 \\ &\omega_{55}=0 \end{aligned} \tag{3.6}$$

We put $\lambda=2$ in (2.16). By applying (3.6) we write down coefficients of the connection ω_{AB}:

$$\begin{aligned}
&\Gamma^{\alpha}_{\beta\gamma}=\bar{\Gamma}^{\alpha}_{\beta\gamma} \\
&\Gamma^{5}_{\beta\gamma}=F_{\beta\gamma}+L_{\beta\gamma}, \qquad \Gamma^{5}_{\beta 5}=Z_{\beta} \\
&\Gamma^{\alpha}_{\beta 5}=F^{\alpha}{}_{\beta}+K^{\alpha}{}_{\beta}, \qquad \Gamma^{\alpha}_{55}=Z^{\alpha} \\
&\Gamma^{\alpha}_{5\gamma}=F^{\alpha}{}_{\gamma}+L^{\alpha}{}_{\gamma}, \qquad \Gamma^{5}_{5\alpha}=\Gamma^{5}_{55}=0
\end{aligned} \tag{3.7}$$

By using (3.7) we define an equation of geodetic line

$$u^{B}\nabla_{B}u^{A}=0 \tag{3.8}$$

in the following form:

$$\begin{aligned}
&\frac{\bar{D}u^{\alpha}}{dt}+\left(2F^{\alpha}{}_{\beta}+K^{\alpha}{}_{\beta}+L^{\alpha}{}_{\beta}\right)u^{\beta}u^{5}=0 \\
&\frac{du^{5}}{dt}+L_{\beta\gamma}u^{\beta}u^{\gamma}+Z_{\beta}u^{\beta}u^{5}=0
\end{aligned} \tag{3.9}$$

where vector u^{α} is normalized

$$g_{\alpha\beta}u^{\alpha}u^{\beta}=1 \tag{3.10}$$

In general, equations (3.9) do not have an integral of motion $u^{5}=\text{const}$. They will have it only when

$$L_{(\beta\gamma)}=0 \quad \text{and} \quad Z_{\alpha}=0 \tag{3.11}$$

Thus we obtain

$$\frac{\bar{D}u^{\alpha}}{dt}+\left(2F^{\alpha}{}_{\beta}+K^{\alpha}{}_{\beta}+L^{\alpha}{}_{\beta}\right)u^{\beta}u^{5}=0, \qquad u^{5}=\text{const} \tag{3.12}$$

where $L_{\alpha\beta}=-L_{\beta\alpha}$. So the condition (3.11) must be added to (3.6) and (3.7). This is forced by a physical necessity to keep q/m_0 constant for a test particle along a world line. It corresponds to the well-known interpretation of u^5 in the classical Klein–Kaluza theory. To be in line with the conventional interpretation of geodetics in that theory we put

$$2H^{\alpha}{}_{\beta}=2F^{\alpha}{}_{\beta}+K^{\alpha}{}_{\beta}+L^{\alpha}{}_{\beta} \tag{3.13}$$

and regard $H^{\alpha}{}_{\beta}$ as a measure of strength of electromagnetic field acting on a test particle. Thus the equations (3.12) become

$$\frac{\overline{D}u^{\alpha}}{dt}+2u^5H^{\alpha}{}_{\beta}u^{\beta}=0, \qquad u^5=\text{const} \tag{3.12a}$$

Hence formally, they do not differ from (2.49)

3.3. Geometry of Manifold *P*. Using (3.13), (3.11), and (3.6) we get

$$\begin{aligned}
\omega_{\alpha\beta}&=\pi^*(\overline{\omega}_{\alpha\beta})+\pi^*\left[H_{\alpha\beta}-\tfrac{1}{2}(L_{\alpha\beta}-K_{\alpha\beta})\right]\theta^5\\
\omega_{\alpha5}&=-\omega_{5\alpha}=\pi^*\left\{\left[H_{\alpha\gamma}+\tfrac{1}{2}(L_{\alpha\gamma}-K_{\alpha\gamma})\right]\overline{\theta}^{\gamma}\right\}\\
\omega_{55}&=0
\end{aligned} \tag{3.14}$$

Now we calculate forms of torsions of ω_{AB}:

$$\Theta^A=D\theta^A \tag{3.15}$$

Applying (3.14) and (3.15) we obtain

$$\begin{aligned}
\Theta^{\mu}&=\pi^*(\overline{\Theta}^{\mu})+\pi^*\left[(L^{\mu}_{\beta}-K^{\mu}_{\beta})\overline{\theta}^{\beta}\right]\wedge\theta^5\\
\Theta^5&=\pi^*\left(L_{\gamma\beta}\overline{\theta}^{\beta}\wedge\overline{\theta}^{\gamma}\right)
\end{aligned} \tag{3.16}$$

where $\overline{\Theta}^{\mu}$ is a torsion of $\omega_{\alpha\beta}$ on E. It is worth noticing that the form Θ^5 is horizontal, so it does not depend on a choice of section of bundle P (choice of gauge). The horizontality is due to the vanishing of vector Z_{α}, for we have in general

$$\Theta^5=\pi^*\left(L_{\alpha\beta}\overline{\theta}^{\beta}\wedge\overline{\theta}^{\alpha}\right)-\pi^*\left(Z_{\beta}\overline{\theta}^{\beta}\right)\wedge\theta^5 \tag{3.17}$$

Now, let us turn back to the equation (3.12a). It is obvious that the test particle is sensitive only to the sum of tensors $K_{\alpha\beta}$ and $L_{\alpha\beta}$. Nevertheless the difference $K_{\alpha\beta}-L_{\alpha\beta}$ appears both in formula (3.13) and in the first one of (3.16). Because of this it seems natural to assume

$$K_{\alpha\beta}=L_{\alpha\beta} \tag{3.18}$$

Thanks to (3.18) forms of torsion of connection become horizontal (they do not depend on a choice of gauge). The equation of geodetics takes the shape

(3.12a), but with $H_{\alpha\beta}=F_{\alpha\beta}+K_{\alpha\beta}$. We also get

$$\Theta^{\mu}=\pi^{*}(\overline{\Theta}^{\mu}) \tag{3.19}$$

$$\Theta^{5}=\pi^{*}\left(K_{\alpha\beta}\bar{\theta}^{\beta}\wedge\bar{\theta}^{\alpha}\right) \tag{3.20}$$

Now it is clear that we add to (3.1) one more assumption, viz.,

$$\Theta^{A}=\operatorname{hor}\Theta^{A} \tag{3.21}$$

This is a simple geometrical condition having some physical motivation. This condition leads to a separation of torsion of space-time from torsion related to the fifth dimension and makes possible a correspondence to the Einstein–Cartan theory. For instance we would not obtain the relation between spin and torsion known from the Einstein–Cartan theory if $Z_{\alpha}\neq 0$. For an arbitrary gauge group G the horizontality of torsion has the identical motivation. When G is non-Abelian the condition (3.1) must be modified, and it will be discussed in Section 6. Inserting (3.18) into (3.14) we obtain

$$\begin{aligned}
&\omega_{\alpha\beta}=\pi^{*}(\bar{\omega}_{\alpha\beta})+\pi^{*}(H_{\alpha\beta})\theta^{5}\\
&\omega_{\alpha 5}=-\omega_{5\alpha}=\pi^{*}\left(H_{\alpha\gamma}\bar{\theta}^{\gamma}\right)\\
&\omega_{55}=0
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&\Gamma^{\alpha}_{\beta\gamma}=\overline{\Gamma}^{\alpha}_{\beta\gamma}, \qquad \Gamma^{5}_{\beta\gamma}=H_{\beta\gamma}\\
&\Gamma^{\alpha}_{\beta 5}=\Gamma^{\alpha}_{5\beta}=H^{\alpha}{}_{\beta}
\end{aligned} \tag{3.23}$$

The rest of the coefficients are equal to zero. Observe that formulas (3.22) and (3.23) formally look like formulas (2.16) and (2.24) from the classical Klein–Kaluza theory. A torsion, thanks to the condition of horizontality, becomes independent on a section. Thus this torsion (especially Θ^{5}), similarly to curvature of connection of the principal bundle P, will have an absolute character of the same type as the strength of the electromagnetic field. In Section 6 we shall discuss in a similar context a form Θ with values in Lie's algebra of G.

Now let us calculate a two-form of curvature of ω_{AB}; using (3.22) and (1.12) we obtain

$$\begin{aligned}
&\Omega^{\alpha}{}_{\beta}=\bar{\Omega}^{\alpha}{}_{\beta}+\bar{D}H^{\alpha}{}_{\beta}\wedge\theta^{5}+\left(H^{\alpha}{}_{\beta}F_{\mu\nu}+H^{\alpha}{}_{\mu}H_{\beta\nu}\right)\bar{\theta}^{\mu}\wedge\bar{\theta}^{\nu}\\
&\Omega^{5}{}_{\beta}=\bar{D}\left(H_{\beta\alpha}\Theta^{\alpha}\right)+\left(H_{\gamma\mu}H^{\gamma}{}_{\beta}\Theta^{\mu}\right)\wedge\Theta^{5}=-\Omega_{\beta}{}^{5}\\
&\Omega^{5}{}_{5}=0
\end{aligned}\tag{3.24}$$

We deleted horizontal lift π^* in formulas (3.24). Defining forms

$$R^{AB}{}_{C}=R^{AB}{}_{CD}\theta^{C}\tag{3.25}$$

in a similar way as in the paper by Trautman (1972), we easily conclude them from formulas (3.24):

$$\begin{aligned}
&R^{\alpha\beta}{}_{\gamma}=\bar{R}^{\alpha\beta}{}_{\gamma}+\bar{\nabla}_{\gamma}H^{\alpha\beta}\theta^{5}+2H^{\alpha\beta}F_{\gamma\nu}\bar{\theta}^{\nu}+2H^{\alpha}_{[\gamma}H^{\beta}_{\nu]}\bar{\theta}^{\nu}\\
&R^{\alpha\beta}{}_{5}=-\bar{D}H^{\alpha\beta}\\
&R^{5\beta}{}_{\gamma}=2\left(\bar{\nabla}_{[\gamma}H^{\beta}_{\alpha]}+H^{\beta\rho}\bar{Q}_{\rho\gamma\alpha}\right)\bar{\theta}^{\alpha}+H_{\delta\gamma}H^{\delta\beta}\theta^{5}\\
&R^{5\beta}{}_{5}=-H_{\gamma\mu}H^{\gamma}_{\beta}\bar{\theta}^{\mu}
\end{aligned}\tag{3.26}$$

The rest of forms R equal zero. We also calculate forms Q^{A}_{B} as in the Einstein–Cartan theory (Trautman, 1972)

$$Q^{A}{}_{B}=Q^{A}{}_{BC}\theta^{C}\tag{3.27}$$

Using (3.27), (3.19), and (3.20) we obtain

$$\begin{aligned}
&Q^{\alpha}{}_{\gamma}=\bar{Q}^{\alpha}{}_{\gamma}\\
&Q^{5}{}_{\gamma}=-2K_{\gamma\alpha}\theta^{\alpha}
\end{aligned}\tag{3.28}$$

The rest of the forms of Q equal zero.

In formulas (3.26) and (3.28) we have omitted lift π^*. In these formulas forms $\bar{R}^{\alpha\beta}{}_{\gamma}$ and $\bar{Q}^{\alpha}{}_{\gamma}$ mean analogical forms on E.

3.4. Variational Principle. Field Equations. As in Section 2, we formulate a variational principle for a scalar of curvature K constructed for

ω_{AB}:

$$K=\tfrac{1}{2}\eta_{AB}\wedge\Omega^{AB} \tag{3.29}$$

We vary K with respect to ω_{AB}, γ_{AB}, and θ^A. Since independent quantities are only $\omega_{\alpha\beta}$, $K_{\alpha\beta}$, $g_{\alpha\beta}$, θ^5, $\bar{\theta}^\alpha$ (due to constraints) we vary K with respect to these quantities. Thus, we have

$$\delta K=\delta\bar{\theta}^A\wedge e_A+\tfrac{1}{2}\delta g_{\alpha\beta}E^{\alpha\beta}-\tfrac{1}{2}\delta\bar{\omega}^\alpha{}_\beta\wedge\bar{p}_\alpha{}^\beta\wedge\theta^5+\delta K^{\alpha\beta}\frac{\delta K}{\delta K^{\alpha\beta}}+\text{exact form} \tag{3.30}$$

Analogous formulas for e_A and $E^{\alpha\beta}$ can be obtained from (2.27) and (2.28) by replacing in them $\tilde{\Omega}^C{}_D$ with the curvature of ω_{AB}. In order to calculate variations with respect to independent components of ω_{AB} we take variation of K with respect to ω_{AB} with constant γ_{AB} and θ^A:

$$(\delta K)\theta^A,\gamma_{AB}=-\tfrac{1}{2}\delta\omega^A{}_B\wedge p_A{}^B \tag{3.31}$$

where

$$p_A{}^B=-D\eta_A{}^B \tag{3.32}$$

By substituting formulas (2.20) and (3.22) into (3.32) we obtain

$$\begin{aligned} p^\alpha{}_\beta&=\bar{p}^\alpha{}_\beta\wedge\Theta^5+2K^\alpha{}_\beta\bar{\eta} \\ p^5{}_\beta&=-p^5{}_\beta=\bar{D}\bar{\eta}_\beta \\ p^5{}_5&=0 \end{aligned} \tag{3.33}$$

Then by inserting (3.33) into (2.31) and varying K with respect to $\bar{\omega}_{\alpha\beta}$ and $K_{\alpha\beta}$ we easily get

$$(\delta K)\theta^A,\gamma_{AB}=-\tfrac{1}{2}\delta\bar{\omega}^\alpha{}_\beta\wedge\bar{p}_\alpha{}^\beta\wedge\theta^5-\delta K^\alpha{}_\beta K_\alpha{}^\beta\bar{\eta}\wedge\theta^5 \tag{3.34}$$

and finally

$$\frac{\delta K}{\delta K^{\alpha\beta}}=-K_{\alpha\beta}\bar{\eta}\wedge\theta^5 \tag{3.35}$$

In formulas (3.33), $\bar{p}^\alpha{}_\beta$ is the Palatini form on manifold E (in $\bar{\omega}_{\alpha\beta}$). It is given by formulas analogous to (3.32) but for quantities defined on E. In the same way as in Section 2 we can calculate forms of e_A and $E^{\alpha\beta}$ by putting into formulas (2.27) and (2.28) the form Ω^C_D given by (3.24). After simple calculations we obtain

$$\begin{aligned} e_\alpha &= \bar{e}_\alpha \wedge \theta^5 + \left(H^{\mu\nu}F_{\mu\nu}\bar{\eta}_\alpha - \tfrac{1}{2}H^{\mu\nu}H_{\mu\nu}\bar{\eta}_\alpha + 2H^{\gamma\mu}F_{\mu\alpha}\bar{\eta}_\gamma\right)\wedge\theta^5 \\ &\quad + \left(\overline{\nabla}_\gamma H^\gamma{}_\alpha + H^\rho{}_\beta \overline{Q}^\beta{}_{\rho\alpha}\right)\bar{\eta} \\ e_5 &= \overline{\nabla}_\beta H^{\beta\gamma}\bar{\eta}_\gamma \wedge \theta^5 + \tfrac{1}{2}\bar{\eta}_{\alpha\beta}\wedge\overline{\Omega}^{\alpha\beta} + H^{\mu\nu}F_{\mu\nu}\bar{\eta} + \tfrac{1}{2}H_{\mu\nu}H^{\mu\nu}\bar{\eta} \qquad (3.36) \\ E^{\alpha\beta} &= \left[\overline{E}^{\alpha\beta} - \left(H^{\mu\alpha}F^\beta_\mu + H^{\mu\beta}F^\alpha_\mu\right)\bar{\eta} + \tfrac{1}{2}g^{\alpha\beta}(F^2 - K^2)\bar{\eta}\right]\wedge\theta^5 \end{aligned}$$

where $F^2 = F_{\mu\nu}F^{\mu\nu}$, $K^2 = K_{\mu\nu}K^{\mu\nu}$. $\overline{E}^{\alpha\beta}, \bar{e}^\alpha$ are, respectively, the symmetrical Einstein form and the Einstein form built from $\overline{\omega}_{\alpha\beta}$ (on manifold E). For simplicity we omitted in formulas (3.33) and (3.36) the horizontal lift π^*. Let us consider the sources. Define a 5-form:

$$\Lambda = \Lambda\left(\gamma_{AB}, \theta^A, \omega^A{}_B, \Psi_a\right) \qquad (3.37)$$

on a manifold P. We vary Λ with respect to independent quantities, i.e., $g_{\alpha\beta}$, $\bar{\theta}^\alpha$, θ^5, $\overline{\omega}_{\alpha\beta}$, $K_{\alpha\beta}$, Ψ_a. Form Λ serves here as a generalized Lagrangian, and we put

$$\underset{\zeta_5}{\mathfrak{L}}\Lambda = 0 \qquad (3.38)$$

By varying with respect to independent variables we obtain

$$\begin{aligned} \delta\Lambda &= \delta\theta^A \wedge t_A + \tfrac{1}{2}\delta g_{\alpha\beta}\pi^*(\overline{T}^{\alpha\beta})\wedge\theta^5 + \tfrac{1}{2}\delta\pi^*\left(\overline{\omega}^\alpha{}_\beta\right)\wedge\pi^*\left(\overline{S}_\alpha{}^\beta\right)\wedge\theta^5 \\ &\quad + \tfrac{1}{2}\delta K^\alpha_\beta \pi^*\left(M^\beta_\alpha\bar{\eta}\right)\wedge\theta^5 + L^a\delta\Psi_a + \text{exact form} \qquad (3.39) \end{aligned}$$

where $\overline{S}^\beta_\alpha$ is a form of spin and $\overline{T}^{\alpha\beta}$ is a tensor of energy-momentum of matter, Ψ_a is a set of "matter" variables, and $L^a = 0$ is an equation of motion of matter. Quantities Ψ_a may be either fields or macroscopic variables, i.e., density, enthalpy, pressure. Later on we shall analyze particular Λ and Ψ_a. It is clear that we have

$$\underset{\zeta_5}{\mathfrak{L}} t_A = 0 \qquad (3.40)$$

and the most general form of t_A is

$$t_\alpha=\pi^*(\bar{t}_\alpha)\wedge\Theta^5+\pi^*(\tfrac{1}{2}i_\alpha\bar{\eta})$$
$$t_5=\pi^*(\tfrac{1}{2}j^\mu\bar{\eta}_\mu)\wedge\theta^5+\pi^*(t\bar{\eta}) \tag{3.41}$$

t_α is a form of energy-momentum of matter and $j=j^\mu\bar{\eta}_\mu$ is a form of current. The quantity i_α is only an auxiliary one and will be eliminated by the Bianchi identity. t is related to a horizontal part of t_5 and eliminated from field equations by Lagrange multipliers.

Notice that from (3.38) we have

$$\Lambda(\gamma_{AB},\theta^A,\omega^A{}_B,\Psi_a)=\mathfrak{L}(g_{\alpha\beta},\bar{\theta}^\alpha,\theta^5,\bar{\omega}^\alpha_\beta,H^\alpha_\beta,\Psi_a) \tag{3.42}$$

where $\mathfrak{L}$ is a 5-form on P. It is easy to understand that

$$\Pi^*(M^{\alpha\beta}\bar{\eta})\wedge\theta^5=\frac{1}{2}\frac{\delta\mathfrak{L}}{\delta H_{\alpha\beta}}=\frac{1}{2}\frac{\delta\mathfrak{L}}{\delta F_{\alpha\beta}}=\frac{1}{2}\frac{\delta\mathfrak{L}}{\delta K_{\alpha\beta}} \tag{3.43}$$

One will see from the definition of $M^{\alpha\beta}$ in (3.43) and (3.39) that we interpret this quantity as electromagnetic polarization. Let us now consider the variational principle

$$\delta\int_V(K-8\pi\Lambda)=0,\qquad V\subset P \tag{3.44}$$

We shall obtain equations from (3.44) by varying $K-8\pi\Lambda$ with respect to independent variables

$$\Theta^A,\qquad g_{\alpha\beta},\qquad \bar{\omega}^\alpha{}_\beta,\qquad K^\alpha{}_\beta,\qquad \Psi_a$$

Constraints $\gamma_{55}=-1,\gamma_{5\alpha}=0$ (discussed in Section 2) allow us in principle to introduce Lagrange's multipliers to eliminate equations connected with the horizontal part of e_5 and t. But we shall not discuss this point in detail in this paper.

Let us now write Cartan's equation, i.e., the equation obtained as a result of varying $K-8\pi\Lambda$ with respect to $\omega^A{}_B$. This yields

$$\bar{p}_{\alpha\beta}=-8\pi\bar{S}_{\alpha\beta}$$
$$K_{\alpha\beta}=-4\pi M_{\alpha\beta} \tag{3.45}$$

The first equation of (3.45) is Cartan's equation known from the Einstein–

Cartan theory. This equation connects torsion with spin. The second equation is a new one which establishes a geometrical interpretation of electromagnetic polarization as torsion related to the fifth dimension. Using this equation we see that

$$H_{\alpha\beta}=F_{\alpha\beta}-4\pi M_{\alpha\beta} \tag{3.46}$$

and we interpret $H_{\alpha\beta}$ as the second tensor of strength of electromagnetic field. Varying $K-8\Pi\Lambda$ with respect to $g_{\alpha\beta}$ we obtain the following equation:

$$\bar{E}^{\alpha\beta}=8\pi\left(\overset{\text{em}}{T^{\alpha\beta}}+\bar{T}^{\alpha\beta}+\pi g^{\alpha\beta}M^2\bar{\eta}\right) \tag{3.47}$$

where

$$M^2=M_{\mu\nu}M^{\mu\nu}$$

$$\overset{\text{em}}{T^{\alpha\beta}}=\frac{1}{8\pi}\left(H^{\mu\alpha}F_{\mu}^{\ \beta}+H^{\mu\beta}F_{\mu}^{\ \alpha}\right)\bar{\eta}-\frac{1}{16\pi}g^{\alpha\beta}F^2\bar{\eta}$$

$$F^2=F_{\mu\nu}F^{\mu\nu}$$

It is worth noting that to obtain (3.47) we have used equations (3.37) and the second equation of (3.45). Variations with respect to $\Theta^A=(\Theta^\alpha,\Theta^5)$ yield the following equations:

$$\bar{e}_\alpha=8\pi\left(\overset{\text{em}}{t_\alpha}+\bar{t}_\alpha+\pi M^2\bar{\eta}_\alpha\right) \tag{3.48}$$

where

$$\overset{\text{em}}{t_\alpha}=\frac{1}{4\pi}\left(H^{\mu\gamma}F_{\mu\alpha}\bar{\eta}_\gamma-\frac{1}{4}F^2\bar{\eta}_\alpha\right)$$

and

$$\bar{\nabla}_\gamma H^{\gamma\beta}=4\pi j^\gamma \tag{3.49}$$

$$\bar{\nabla}_\gamma H_\alpha^\gamma+H^\rho{}_\beta\bar{Q}^\beta{}_{\rho\alpha}=4\pi i_\alpha \tag{3.50}$$

Constraints eliminate an equation in which t appears. The equation (3.50) can be reduced to (3.49) by Bianchi's identities.

Equations (3.47) and (3.48) are equivalent because of identities known from the Einstein–Cartan theory (Trautman, 1972)

$$\bar{E}^{\alpha\beta}=\bar{\theta}_\beta\wedge\bar{e}_\alpha-\tfrac{1}{2}\bar{D}p_{\alpha\beta} \tag{3.51}$$

$$T_{\alpha\beta}=\bar{\theta}_\beta\wedge t_\alpha-\tfrac{1}{2}\bar{D}\bar{S}_{\alpha\beta} \tag{3.52}$$

and the relation between $\overset{\text{em}}{T}{}^{\alpha\beta}$ and $\overset{\text{em}}{t}_\alpha$. Equations (3.47) and (3.48) are equations of gravitation in Einstein–Cartan theory. Equation (3.49) is the second pair of Maxwell equations for the second tensor of strength of electromagnetic field $H_{\alpha\beta}$. Derivatives in (3.49) are covariant derivatives with respect to $\bar{\omega}^\alpha{}_\beta$ (with torsion). The tensor $\overset{\text{em}}{T}{}^{\alpha\beta}$ is a symmetrized form of the tensor $\overset{\text{em}}{t}{}^{\alpha\beta}$, where

$$\overset{\text{em}}{t}{}^{\alpha}=\overset{\text{em}}{t}{}^{\alpha\beta}\bar{\eta}_\beta$$

3.5. Interpretations of $M_{\alpha\beta}$ and $K_{\alpha\beta}$. Comparison to W. Israel's Results. Now we give an interpretation of the equations achieved in Section 3.4 and quantities introduced there. First we notice that quantities $\overset{\text{em}}{T}{}^{\alpha\beta}$ and $\overset{\text{em}}{t}_\alpha$ introduced in (3.47) and (3.48) may be regarded as forms of energy-momentum of electromagnetic field symmetrical and nonsymmetrical, respectively. $H_{\alpha\beta}$ is regarded as the second tensor of electromagnetic field.

$\overset{\text{em}}{t}_\alpha$ is a form of energy-momentum reported in Israel's papers (Bailey and Israel, 1975; Israel, 1977, 1974) and $\overset{\text{em}}{T}{}^{\alpha\beta}$ a symmetrized form of Israel's tensor.

The second equation (3.45) defines a relation between torsion in fifth dimension and electromagnetic polarization of matter. It is an illustration of Einstein's postulate that matter quantities are on the right-hand side of equations and geometrical ones on the left-hand side of them. Electromagnetic polarization becomes the source of torsion. This equation is algebraical, so torsion is present and nonvanishing only when electromagnetic polarization does not vanish hence a torsion may not be zero only when matter is present. This equation is of the same type as the equation relating spin to torsion in Einstein–Cartan theory and it has been achieved in a similar way. Let us now introduce a two-form of electromagnetic polarization

$$M=\tfrac{1}{2}M_{\mu\nu}\bar{\theta}^\mu\wedge\bar{\theta}^\nu \tag{3.53}$$

and a two-form of the second tensor of strength of electromagnetic field

$$H=\tfrac{1}{2}H_{\alpha\beta}\bar{\theta}^{\alpha}\wedge\bar{\theta}^{\beta} \tag{3.54}$$

Taking into account the second equation of (3.45) and (3.20) we get

$$\Omega-\tfrac{1}{2}\Theta^{5}=\pi^{*}(H) \tag{3.55}$$

Equation (3.55) establishes a relation between "matter" and geometrical quantities. Actually it states that form H, a "matter" quantity, is a source of a geometrical quantity. The equation (3.55) is an analogon of the equation (2.8) present in the classical Klein–Kaluza theory.

In Section 6 we shall obtain a generalization of equation (3.55) when the gauge group is not just $U(1)$ and may be non-Abelian.

Instead of (3.45) we can write

$$\Theta^{5}=\pi\pi^{*}(M) \tag{3.56}$$

Now let us turn back to equations (3.47) and (3.48). Define the symbols

$$\overset{\text{em}}{T^{\alpha\beta}}+\pi g^{\alpha\beta}M^{2}=\overset{\text{em.tot}}{T^{\alpha\beta}}$$

$$\overset{\text{em}}{t_{\alpha}}+\pi M^{2}\bar{\eta}_{\alpha}=\overset{\text{em.tot}}{t_{\alpha}} \tag{3.57}$$

and

$$\overset{\text{tot}}{T^{\alpha\beta}}=\overset{\text{em.tot}}{T^{\alpha\beta}}+\bar{T}^{\alpha\beta}$$

$$\overset{\text{tot}}{t_{\alpha}}=\overset{\text{em.tot}}{t_{\alpha}}+\bar{t}_{\alpha} \tag{3.58}$$

Then equations (3.47) and (3.48) take a form

$$\bar{E}^{\alpha\beta}=8\pi\overset{\text{tot}}{T^{\alpha\beta}}$$

$$\bar{e}_{\alpha}=8\pi\overset{\text{tot}}{t_{\alpha}} \tag{3.59}$$

Thus equations (3.59) are the Einstein–Cartan equations with the sum of the following energy-momentum tensors: of matter, of electromagnetic field (of Israel type with polarization of matter taken into account), plus a new additional pressure-type component. We shall analyze this component $-\pi M^2 g_{\alpha\beta}$ and estimate its role. We have introduced symbols $\overset{\text{em.tot}}{T^{\alpha\beta}}$ and $\overset{\text{em.tot}}{t_\alpha}$ because the additional component with $M^2 = M_{\mu\nu} M^{\mu\nu}$ may be treated as a kind of electromagnetic interaction. This is a contact interaction (electromagnetic polarization)×(electromagnetic polarization). This interaction appears due to torsion related to the fifth dimension and is of the same type and origin as a (spin)×(spin) interaction in the Einstein–Cartan theory.

The (spin)×(spin) interaction in the Einstein–Cartan theory is considered to be of gravitational origin. So it seems natural that we consider the new interaction the additional electromagnetic interaction. Perhaps this interaction may be related to a nonlinear electrodynamics. In Section 4 we shall discuss some consequences of the existence of this interaction.

In their work Bailey and Israel (1975) analyzed, in a phenomenological way, a theory of spinning particles, electrically charged with magnetic moment. The procedure developed by us in this chapter geometrizes W. Israel's results and leads to some new additional effects. Namely, the new features of our theory when compared with Israel's model are the appearance of the component $\pi M^2 g_{\alpha\beta}$ and the fact that the second pair of Maxwell equations are written by aid of covariant derivatives with respect to connection with torsion.

Israel's theory does not use Einstein–Cartan theory as a gravitational theory. For this reason many interesting geometrical relationships cannot be obtained in this theory. It seems to us that both the role of torsion and the program of geometrization of matter quantities were underestimated in the Israel approach to the problem.

3.6. Bianchi's Identities. Conservation Laws. We obtain the contracted Bianchi's identities from the Bianchi identity (1.16)

$$
\begin{aligned}
De_A &= Q^B{}_A \wedge e_B - \tfrac{1}{2} R^{BC}{}_A \wedge p_{BC} \\
Dp_{AB} &= e_B \wedge \theta_A - e_A \wedge \theta_B
\end{aligned}
\tag{3.60}
$$

Substituting equations (3.36), (3.33), and field equations (3.45), (3.48), (3.49) into the second of (3.60) and taking formulas (3.26) and (3.28) into account we get the angular momentum conservation:

$$
\bar{D}\bar{S}_{\alpha\beta} = \bar{\theta}_\alpha \wedge \overset{\text{tot}}{t_\beta} - \bar{\theta}_\beta \wedge \overset{\text{tot}}{t_\alpha}
\tag{3.61}
$$

and the identity

$$4\pi(j_\alpha+i_\alpha)=\tfrac{1}{2}H^{\delta}{}_{\alpha}\bar{S}_{\delta}+H^{\gamma\delta}\bar{S}_{\gamma\delta\alpha} \tag{3.62}$$

where $\bar{S}_\delta=\bar{S}^{\gamma}{}_{\delta\gamma}$. The last identity is not a new conservation law. It only establishes the relation between j_α and i_α. In this way it eliminates equation (3.50), reducing it to (3.49). Finally we have got the equation (3.49) as the only equation of electromagnetic field (the second pair of Maxwell equations).

This equation is written by aid of covariant derivatives with respect to a connection with torsion. We can write them as Riemannian derivatives on E and spin. We have

$$\bar{\nabla}_{\beta}H^{\gamma\beta}=\bar{\tilde{\nabla}}_{\beta}H^{\gamma\beta}+\pi\bar{S}^{\gamma}{}_{\beta\alpha}H^{\beta\alpha} \tag{3.63}$$

Thus equation (3.49) is equivalent to

$$\bar{\tilde{\nabla}}_{\beta}H^{\gamma\beta}=4\pi\overset{\text{tot}}{j^{\gamma}} \tag{3.64}$$

where

$$\overset{\text{tot}}{j^{\gamma}}=j^{\gamma}+j_i^{\gamma}=j^{\gamma}-\tfrac{1}{4}\bar{S}^{\gamma}{}_{\beta\alpha}H^{\beta\alpha} \tag{3.65}$$

The internal current $j_i^\gamma=-\frac{1}{4}\bar{S}^{\gamma}{}_{\beta\alpha}H^{\beta\alpha}$ satisfies the generalized "Ohm's law" (proportionality to strength of field). In Section 4 we estimate the contribution of j_i^γ to $\overset{\text{tot}}{j^{\gamma}}$, and in Section 5 we find certain other implications of the existence of this current. Here we only notice that j_i^γ is of gravitational–electromagnetic nature. Thanks to this, both gravitational and electromagnetic fields are more strongly interrelated than in classical Klein–Kaluza theory.

Now let us turn back to the first equation of (3.60). We get from it

$$\bar{D}\overset{\text{tot}}{t_{\alpha}}=\bar{Q}^{\beta}_{\alpha}\wedge\overset{\text{tot}}{t_{\beta}}-\tfrac{1}{2}\bar{R}^{\beta\gamma}{}_{\alpha}\wedge\bar{S}_{\beta\gamma} \tag{3.66}$$

i.e., conservation law of energy-momentum in the Einstein–Cartan theory for total energy matter and electromagnetic and gravitational fields. We also obtain the continuity equation

$$d\overset{\text{tot}}{j}=0,\qquad \overset{\text{tot}}{j}=\overset{\text{tot}}{j^{\gamma}}\bar{\eta}_{\gamma} \tag{3.67}$$

i.e., conservation of a charge.

Thus we have achieved the following laws: energy-momentum conservation, conservation of charge, and angular momentum conservation.

3.7. Geodetic Lines. Interpretation. Equations of geodetic line (3.12a) on manifold P could be interpreted as an equation of motion of a test particle in both electromagnetic and gravitational fields. A particle without spin is believed, in the Einstein–Cartan theory, to move along the Riemannian geodetic line. But the equation (3.12a) cannot be regarded as an equation of motion of a spinning particle (there is no component with a magnetic moment). So, in our theory the equation of motion of a test particle is

$$\frac{\tilde{\bar{D}} u^{\alpha}}{dt} + 2u^5 H^{\alpha}_{\beta} u^{\beta} = 0, \qquad u^5 = \text{const} \tag{3.68}$$

It is an equation of geodetic line on manifold P with respect to the connection $\hat{\omega}_{AB}$ which differs from ω_{AB} only in that

$$\hat{\omega}_{\alpha\beta} = \pi^*\left(\hat{\bar{\omega}}_{\alpha\beta}\right) + H_{\alpha\beta}\Theta^5 \neq \omega_{\alpha\beta}$$

In Lorentz's force term in the equation (3.68) there is the tensor $H_{\alpha\beta} = F_{\alpha\beta} + K_{\alpha\beta}$ that describes a total electromagnetic field with a polarization of matter. Thus, the equation (3.68) is a generalization of both the equation of motion of a spinless particle in the Einstein–Cartan theory and equation (2.51) from the Klein–Kaluza theory.

3.8. Example. Vector Field Charged on *P*. Let us define a vector field on P

$$\begin{gathered} W_A(pg_1) = \sigma\left(g_1^{-1}\right) \cdot W_A(p) \\ W_A^*(pg_1) = W^*(p)\sigma(g_1) \\ p = (x, g) \in P, \qquad g, g_1 \in U(1) \\ \sigma: U(1) \to GL(5, \mathbb{C}) \end{gathered} \tag{3.69}$$

For any section $f: E \to P$ we have

$$f' W_A = \left(W^f_{\alpha}, \varphi^f\right) \tag{3.70}$$

W_{α} and φ are a vector and scalar field, respectively, defined on E. Interaction among field W_A and gravitational and electromagnetic fields described by the Klein–Kaluza theory with torsion is introduced by taking in Lagrange form derivatives in the form

$$\mathfrak{D}W_A = \operatorname{hor} DW_A$$
$$\mathfrak{D}W_A^* = \operatorname{hor} DW_A^* \tag{3.71}$$

Thus we have

$$\mathfrak{D}W_\alpha = \overline{\mathfrak{D}}W_\alpha - \varphi H_{\alpha\gamma}\theta^\gamma$$
$$\mathfrak{D}W_5 = d_1\varphi - W^\beta H_{\beta\gamma}\theta^\gamma \tag{3.72}$$
$$\mathfrak{D}W^{*\alpha} = \overline{\mathfrak{D}}W^{*\alpha} - \varphi^* H^\alpha_\beta \theta^\beta \tag{3.73}$$
$$\mathfrak{D}W_5^* = -\left(d_1\varphi^* - W^{*\beta}H_{\beta\gamma}\theta^\gamma\right)$$

These formulas have been obtained with the aid of (3.22). $\overline{\mathfrak{D}}$ is an exterior differential with respect to both $\pi^*(\omega^\alpha_\beta)$ and α "gauge", e.g., we have

$$\overline{\mathfrak{D}}W_\alpha = \operatorname{hor}\overline{D}\,W_\alpha \tag{3.74}$$

Now let us analyze Lagrange 5-form

$$\Lambda = \kappa\,\mathfrak{D}W_A \wedge (\mathfrak{D}\,W^{A*}), \qquad \kappa = \text{const} \tag{3.75}$$

Substituting formulas (3.72) and (3.73) into (3.75) we easily find

$$\begin{aligned}\Lambda = {} & \kappa\pi^*\left[\overline{\mathfrak{D}}W_\alpha \wedge (\overline{\mathfrak{D}}W^{*\alpha})^\# - d_1\varphi \wedge (d_1\varphi^*)^\#\right] \wedge \theta^5 + \\ & + \kappa H^{\gamma\alpha}\Big[\overline{\varphi}^*\left(\overline{\nabla}_\gamma - iqA_\gamma\right)W_\alpha + \varphi\left(\overline{\nabla}_\gamma + iqA_\gamma\right)W_\alpha^* + \\ & + \left(\partial_\gamma - iqA_\gamma\right)\varphi^* W_\alpha^* + \left(\partial_\gamma + iqA_\gamma\right)\varphi^* W_\alpha\Big] \wedge \Pi^*(\overline{\eta}) \wedge \theta^5 + \\ & + \kappa\left(|\varphi|^2 H_{\alpha\beta}H^{\alpha\beta} - W^\alpha W^{*\beta}H^\gamma_\alpha H_{\beta\gamma}\right)\pi^*(\overline{\eta}) \wedge \theta^5 \end{aligned} \tag{3.76}$$

In formula (3.76), apart from the sum of Lagrangians of both scalar and vector fields, the additional interferential components appeared. According to our rules we calculate variations with respect to independent variables of a connection obtaining both spin and electromagnetical polarization.

We have

$$M_{\alpha\beta}=\kappa\Big[\varphi^*\big(\overline{\nabla}_{\alpha}-iqA_{[\alpha)}\big)W_{\beta]}+\varphi\big(\overline{\nabla}_{[\alpha}+iqA_{[\alpha)}\big)W^*_{\beta]}+\big(\partial_{[\alpha}-iqA_{[\alpha)}\big)\varphi W^*_{\beta]}+$$
$$+\big(\partial_{[\alpha}+iqA_{[\alpha)}\big)\varphi^* W_{\beta]}+2|\varphi|^2 H_{\alpha\beta}-W_{[\alpha}W^{*\gamma}H_{|\gamma|\beta]}-W^*_{[\alpha}W^{\gamma}H_{|\gamma|\beta]}\Big] \tag{3.77}$$

Unfortunately the equation (3.77) is so complicated that a general analysis of it seems to be a really difficult task. So we discuss here only a particular case when $W_\alpha=0$.

jf $W_\alpha=0$; then we have

$$\Lambda=\pi^*\big[-\kappa\, d_1\varphi\wedge(d_1\varphi^*)^{\#}\big]\wedge\theta^5+\kappa|\varphi|^2 H_{\alpha\gamma}H^{\alpha\gamma}\bar{\eta}\wedge\theta^5 \tag{3.78}$$

and

$$M_{\alpha\beta}=2\kappa|\varphi|^2 H_{\alpha\beta} \tag{3.79}$$

From equation (3.45) it follows that

$$K_{\alpha\beta}=-\frac{8\pi\kappa|\varphi|^2}{1+8\pi\kappa|\varphi|^2}F_{\alpha\beta} \tag{3.80}$$

and

$$H_{\alpha\beta}=\frac{F_{\alpha\beta}}{1+8\pi\kappa|\varphi|^2} \tag{3.80a}$$

This particular case has an interesting feature—the form of spin vanishes. Hence the torsion of space-time vanishes as well, but it still is not equal to zero in the fifth dimension. From the equation (3.78) we get a field equation

$$g_{\alpha\beta}(\partial^\alpha-iqA^\alpha)(\partial^\beta-iqA^\beta)=8\varphi\frac{S}{\big(1+8\pi\kappa|\varphi|^2\big)^2} \tag{3.81}$$

where

$$S=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=(B^2-E^2)$$

So, in equations (3.81) has appeared a component with an effective mass depending on electromagnetic field. In some very strong electromagnetic fields such a "mass" could have reasonable values. And we see that in the case just discussed the field causes the creation of a polarization of vacuum (3.80). Notice also that the geometrical quantity W_A defined on P may be regarded as a multiplet of vector W_α and scalar φ fields taken in any gauge. This can be extended to any gauge group G. In Section 5 the spinor field Ψ will be treated in a similar way. We shall use there the derivative $\mathfrak{D}$ as well.

4. APPLICATIONS

In this section we discuss new physical effects appearing in the Klein–Kaluza theory with torsion. We estimate the contribution of a new additional pressure-type component. We shall define a component describing coupling between electromagnetic field and spin in the Einstein–Cartan theory and we shall also arrive at a cosmological model without either initial or final singularities in $R(t)$—the radius of the universe. The equation (3.59) may be written in tensor notation (we turn back to the cgs system):

$$\overline{R}_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}\overline{R}=\frac{8\pi G}{c^4}\left(t_{\alpha\beta}+\overset{\text{em}}{t}_{\alpha\beta}+\pi M^2 g_{\alpha\beta}\right) \tag{4.1}$$

where

$$\overset{\text{em}}{t}_{\alpha\beta}=\frac{1}{4\pi}\left(H^{\mu}_{\alpha}F_{\mu\beta}-\tfrac{1}{4}F^2 g_{\alpha\beta}\right)$$

is a tensor of energy-momentum of electromagnetic field in Israel's form.

We also write the equation (3.45) in the tensor notation

$$\overline{Q}^{\alpha}{}_{\beta\gamma}+\delta^{\alpha}_{\beta}\overline{Q}^{\delta}{}_{\gamma\delta}-\delta^{\alpha}_{\gamma}\overline{Q}^{\delta}{}_{\beta\delta}=\frac{8\pi G}{c^3}\overline{S}^{\alpha}{}_{\beta\gamma} \tag{4.2}$$

By applying formula (4.2) and the relation between the Riemannian connection and Cartanian one on E we obtain the formula (4.1) in the form known from the work of Arkuszewski, Kopczyński, and Ponomariew (1974):

$$\begin{aligned}\tilde{\overline{R}}_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}\tilde{\overline{R}}=\frac{8\pi G}{c^4}\left(t_{\alpha\beta}+\overset{\text{em}}{t}_{\alpha\beta}+\Pi M^2 g_{\alpha\beta}\right)+\frac{4\pi G}{c^3}\tilde{\overline{\nabla}}_{\gamma}\left(\overline{S}^{\gamma}{}_{\alpha\beta}+\overline{S}^{\gamma}{}_{\alpha\beta}+\overline{S}_{\beta\alpha\gamma}\right)\\ -\left(\frac{4\Pi G}{c^3}\right)^2\Big[2\overline{S}_{\alpha\beta\gamma}\overline{S}^{\gamma}+2\overline{S}_{\alpha\gamma\delta}\overline{S}^{\delta\gamma}{}_{\beta}+\overline{S}_{\alpha\gamma\delta}\overline{S}^{\gamma\delta}{}_{\beta}\\ +g_{\alpha\beta}\left(\overline{S}_{\gamma}\overline{S}^{\gamma}-\overline{S}_{\delta\gamma\mu}\overline{S}^{\mu\gamma\delta}-\tfrac{1}{2}\overline{S}_{\delta\gamma\mu}\overline{S}^{\gamma}{}_{\alpha\gamma}\right)\Big]\end{aligned} \tag{4.3}$$

where $\overline{S}_{\alpha}=\overline{S}^{\gamma}{}_{\alpha\gamma}$.

Now let us consider the simple model described by the following formulas:

$$
\begin{aligned}
&t^{\alpha}{}_{\beta}=u^{\alpha}h_{\beta}-p\delta^{\alpha}_{\beta}\\
&\bar{S}^{\alpha}{}_{\beta\gamma}=u^{\alpha}S_{\beta\gamma}, \qquad u^{\beta}S_{\beta\gamma}=0\\
&M_{\alpha\beta}=\rho S_{\alpha\beta}\\
&j_{\alpha}=nqcu_{\alpha}
\end{aligned}
\tag{4.4}
$$

This is a dust model, where h_{β} is a four-vector of enthalpy, u_{α} is a four-vector of velocity, ρ is a gyromagnetic ratio, n is a concentration of dust particles, and q is a charge of a dust particle. Quantities (4.4) must obey the Bianchi identities, i.e., conservations law, in particular angular momentum conservation, which has the form

$$\overline{D}\bar{S}_{\alpha\beta}=\frac{1}{c}\bar{\theta}_{\alpha}\wedge\bar{t}_{\beta}-\frac{1}{c}\bar{\theta}_{\beta}\wedge\bar{t}_{\alpha}-\frac{1}{c}\left(M^{\mu}_{\alpha}F_{\mu\beta}-M^{\mu}_{\beta}F_{\mu\alpha}\right)\bar{\eta} \tag{4.5}$$

Substituting (4.4) into the formula (4.5) we obtain the formula for h_{β}:

$$h_{\beta}=(e+p)u_{\beta}+cu^{\alpha}u^{\gamma}\tilde{\overline{\nabla}}_{\gamma}S_{\beta\alpha}-u^{\gamma}\left(M^{\delta}{}_{\gamma}F_{\delta\beta}-M^{\delta}{}_{\beta}F_{\delta\gamma}\right) \tag{4.6}$$

Inserting (4.6) into the first formula of (4.4) and using the relation between $M_{\alpha\beta}$ and $S_{\alpha\beta}$ we get

$$t_{\alpha\beta}=(e+p)u_{\alpha}u_{\beta}+pg_{\alpha\beta}+cu_{\alpha}u^{\delta}u^{\gamma}\tilde{\overline{\nabla}}_{\gamma}S_{\beta\delta}-\rho u_{\alpha}S^{\delta}{}_{\beta}F_{\delta\gamma}u^{\gamma} \tag{4.7}$$

where $e=t_{\alpha\beta}u^{\alpha}u^{\beta}$ is the density of energy. Applying the formula (4.7) and (4.4) and (4.3) we finally obtain

$$
\begin{aligned}
\tilde{\bar{R}}_{\alpha\beta}-\tfrac{1}{2}\tilde{\bar{R}}g_{\alpha\beta}=\frac{8\pi G}{c^{4}}\Bigg[&\overset{\text{em.vac.}}{t_{\alpha\beta}}+\left(e+p-\frac{4\pi G}{c^{2}}S^{2}\right)u_{\alpha}u_{\beta}+\\
&-\left(p-\frac{2\Pi G}{C^{2}}S^{2}-2\Pi\rho^{2}S^{2}\right)g_{\alpha\beta}+\\
&-C\left(g^{\delta\gamma}+u^{\delta}u^{\gamma}\right)\tilde{\overline{\nabla}}S_{\gamma(\alpha}U_{\beta)}+\rho\left(u_{\alpha}S^{\delta}{}_{\beta}F_{\delta\gamma}u^{\gamma}-S^{\mu}{}_{\alpha}F_{\mu\beta}\right)\Bigg]
\end{aligned}
\tag{4.8}
$$

In the formula (4.8) we have explicit a term of coupling between spin and electromagnetic field. Namely, it is

$$\rho\left(u_\alpha u^\gamma S^\delta{}_\beta F_{\delta\gamma} - S^\mu{}_\alpha F_{\mu\beta}\right) \tag{4.9}$$

Now we conclude that (4.9) is the term that has been searched for in the Einstein–Cartan theory.

$$S^2 = \tfrac{1}{2} S_{\alpha\beta} S^{\alpha\beta}$$

and $\overset{\text{em.vac.}}{t_{\alpha\beta}}$ is a tensor of energy-momentum of electromagnetic field in vacuum.

The additional component (4.9) may influence the evolution of stars with magnetic field. It would be very interesting to analyze this term in relation to a theory of neutron stars. In such a case ρ would be a nuclear magneton. Now let us introduce the symbols

$$p_{\text{eff}} = p - \frac{2\pi G}{c^2} S^2 - 2\pi\rho^2 S^2 = p_{\text{tot}} - \frac{2\pi G}{c^2} S^2$$

$$e_{\text{eff}} = e - \frac{2\pi G}{c^2} S^2 + 2\pi\rho^2 S^2 = e_{\text{tot}} - \frac{2\pi G}{c^2} S^2 \tag{4.10}$$

and

$$p_{\text{tot}} = p - 2\pi\rho^2 S^2$$

$$e_{\text{tot}} = e + 2\pi\rho^2 S^2 \tag{4.11}$$

The formula (4.8) really differs from the analogous formula given in Arkuszewski, Kopczyński, and Ponomariew (1974). The differences are as follows. First of all there is a component giving a coupling between electromagnetic field and spin on the right-hand side of the equation. Secondly there has appeared an additional pressure-type component. This component gives a correction to e and p. However, the correction term to e has a reverse sign from the correction in the Einstein–Cartan theory. Now we estimate the contribution of this new correction in relation to that known from the Einstein–Cartan theory. To do this we assume that our fluid is a nuclear fluid, thus

$$\rho = \frac{q}{m_p c}$$

Taking the ratio of these two corrections we have

$$\frac{\text{new component connected to the fifth dimension}}{\text{additional component known in Einstein–Cartan theory}} = \frac{c^2\rho^2}{G}$$

$$= \frac{q^2}{m^2{}_pG} \simeq 10^{36}$$

An interesting fact is that the ratio $q^2/m^2{}_pG$ is simply equal to the ratio of electric interaction of two protons to their gravitational interaction. Now let us estimate a density for which this correction will be comparable to the density of energy e.

We have

$$S \simeq \tfrac{1}{2}n\hbar, \qquad 2\pi\rho^2 S^2 \simeq \frac{q^2 n^2 \hbar^2}{4m_p^2 c^2}, \qquad e \simeq m_p c^2 n$$

Assuming that

$$e \simeq m_p c^2 n \simeq \frac{q^2 n^2 \hbar^2}{4mp^2 c^2}$$

we obtain

$$n_1 \simeq 10^{43}\ \text{cm}^{-3}$$

But it is known that the correction to e in the Einstein–Cartan theory is comparable to e for a concentration

$$n_2 \simeq 10^{79}\ \text{cm}^{-3}$$

Concentrations n_1 and n_2 correspond to matter densities:

$$\rho_1 = 10^{18}\ \text{g cm}^{-3}, \qquad \rho_2 = 10^{54}\ \text{g cm}^{-3}$$

The same order of concentration as n_1 is possible in the center of a neutron star. So this correction should play a certain role in the evolution of a neutron star, gravitational collapse, and in cosmological models. Notice also that for an electron gas we have

$$n_1 = 10^{33}\ \text{cm}^{-3}$$

Now we turn to estimating the contribution of additional current j_i^γ to total current. Let us consider the equation (3.64)

$$\tilde{\overline{\nabla}}_\beta H^{\gamma\beta}=\frac{4\pi}{c}\overset{\text{tot}}{j^\gamma} \tag{4*}$$

where

$$\overset{\text{tot}}{j^\gamma}=u^\gamma\left(nqc-\frac{G}{4c^2}F^{\beta\alpha}S_{\beta\alpha}+\frac{2\pi G}{c^2}\rho S^2\right) \tag{4.12}$$

The current $\overset{\text{tot}}{j^\gamma}$ may be regarded as a convective current by introducing

$$q_{\text{eff}}=q-\frac{G}{4nc^3}F^{\beta\alpha}S_{\beta\alpha}+\rho\frac{2\pi G}{nc^3}S^2$$

$$\overset{\text{tot}}{j^\gamma}=ncq_{\text{eff}}u^\gamma \tag{4.13}$$

We estimate now a density for which the second correction to q_{eff} is comparable to q. Putting $S\simeq\frac{1}{2}n\hbar$ we obtain

$$\frac{\pi Gq\hbar^2 n}{2m_p c^4}\simeq q$$

from which it follows that

$$n\simeq\frac{2m_p c^4}{\pi G\hbar^2}\simeq 10^{78}\text{ cm}^{-3}$$

Notice that n is the same order as the density n_2 in the Einstein–Cartan theory. The first correction to q_{eff} depends on the electromagnetic field and gives a certain "electromagnetic structure" of an effective charge, but is very small and unobservable in normal situations. Now let us consider once more the equation (4.8). It seems quite interesting to analyze cosmological models based on it. Let us adapt the already existing Kopczyński models to estimate the role of an additional component related to the square of electromagnetic polarization. We neglect components connected with derivatives of spin and we include electromagnetic field in the matter term.

We also put a metric tensor in the Robertson–Walker form

$$ds^2=c^2\,dt^2-R^2(t)\frac{dx^2+dy^2+dz^2}{1+\frac{1}{4}K\left(x^2+y^2+z^2\right)} \tag{4.14}$$

where $K=0, \pm 1$. The equation (4.8) takes the form

$$\tilde{\bar{R}}_{\alpha\beta}-\tfrac{1}{2}\tilde{\bar{R}}g_{\alpha\beta}=\frac{8\pi G}{c^4}\left[(q_{\text{eff}}+p_{\text{eff}})u_\alpha u_\beta-p_{\text{eff}}g_{\alpha\beta}\right] \tag{4.15}$$

We assume in addition $\sigma=S_{12}\neq 0$ and $S_{\mu\nu}=0$ when $\mu+\nu\neq 3$. It is easy to see from (4.5) the spin conservation law as in the Einstein–Cartan theory

$$d(S\bar{u})=0 \tag{4.16}$$

where

$$S=\left(\tfrac{1}{2}|S_{\alpha\beta}S^{\alpha\beta}|\right)^{1/2}$$

Inserting (4.16) into our cosmological model we obtain

$$S^2=\tfrac{1}{2}S_{\alpha\beta}S^{\alpha\beta}=\sigma^2$$

$$\tfrac{4}{3}\pi\sigma R^3=\mu=\text{const} \tag{4.17}$$

We get Friedmann's equations after substituting (4.17) and (4.14) into (4.15)

$$\frac{8\pi G}{c^4}p_{\text{eff}}=-2\frac{\ddot{R}}{c^2R^2}-\frac{\dot{R}^2}{c^2R^2}-\frac{K}{R^2}$$

$$\frac{8\pi G}{c^4}e_{\text{eff}}=3\frac{\dot{R}^2}{c^2R^2}+\frac{3K}{R^2} \tag{4.18}$$

Assuming that $p=0$ and substituting (4.10) we finally get

$$\frac{2\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}-\frac{9\mu^2G}{c^2R^6}\left(\frac{G}{c^2}+2\pi\rho^2\right)+\frac{Kc^2}{R^2}=0 \tag{4.19}$$

and

$$\frac{8\pi G}{c^4}e_{\text{tot}}=\frac{3\dot{R}^2}{R^2c^2}+\frac{9\mu^2G}{R^6c^4}+\frac{3K}{R^2} \tag{4.20}$$

where e_{tot} is given by (4.11). Using (4.17) we have

$$e_{\text{tot}}=e+\frac{9\mu^2\rho^2}{8\pi R^6} \tag{4.21}$$

Now let us examine equation (4.19) and find a first integral of (4.19)

$$\frac{R^2}{2}-\frac{MG}{R}+\frac{3\mu^2 G}{2R^4c^2}\left(\frac{G}{c^2}+2\pi\rho^2\right)=-\frac{Kc^2}{2} \tag{4.22}$$

Where M is a constant of integration. We shall find its interpretation by substituting (4.22) into (4.20) and using (4.21).

In effect we have

$$\frac{4\pi}{3}R^3 e_{\text{tot}}=Mc^2 \tag{4.23}$$

So we see that M has an interpretation of the total mass of the universe. Notice that it is just the energy e_{tot} that is conserved but not e. The last fact is obvious for we have achieved the conservation law for $\overset{\text{tot}}{t}_\alpha$. Now let us consider the equation (4.22). Its solution is

$$t=\int_{R_{\min}}^{R} dr\Big/\left[-Kc^2+\frac{2MG}{r}-\frac{3\mu^2 G}{c^2 r^4}\left(\frac{G}{c^2}+2\pi\rho^2\right)\right]^{1/2} \tag{4.24}$$

as in Kopczyński's work (1973). We perform integration for $K=0$ and get

$$R(t)=\left[\frac{GMt^2}{2}+\frac{3\mu}{2Mc^2}\left(\frac{G}{c^2}+2\pi\rho^2\right)\right]^{1/3} \tag{4.25}$$

We obtain the three types of models of the universe with minimal radii $R_{\min}\neq 0$ elliptical, flat, hyperbolical. From the equation (4.22) we obtain for turning points $\dot{R}=0$

$$KR_{\text{ex}}^4-\frac{2MG}{c^2}R_{\text{ex}}^2+\frac{3\mu^2 G}{c^4}\left(\frac{G}{c^2}+2\pi\rho^2\right)=0 \tag{4.26}$$

In the elliptical case solutions (4.26) give both minimal and maximal radius of the universe. For the flat case $K=0$ we get

$$R_{\min}=R(0)=\left[\frac{2\mu^2}{2Mc^2}\left(\frac{G}{c^2}+2\pi\rho^2\right)\right]^{1/3} \tag{4.27}$$

Substituting $M=m_p N$, $N\simeq 10^{80}$ and $\mu=\frac{1}{2}\hbar N$ (where N is the number of nucleons in universe) into (4.27), we have

$$R_{\min}\simeq 10^{12}\ \text{cm} \tag{4.28}$$

That is 12 orders more than in the models of Einstein–Cartan theory. Estimation (4.28) will also be true for both elliptical and hyperbolical models when the following condition is fulfilled:

$$\frac{c^2}{2MG}\left[\frac{3\mu^2}{2Mc^2}\left(\frac{G}{c^2}+2\pi\rho^2\right)\right]^{1/3}\ll 1 \tag{4.29}$$

as in Kopczyński's work (1973).

Now let us estimate the contribution of the component M^2 to the density of energy for the contemporary epoch, i.e., when $R(t_0)\simeq 10^{36}$ cm. Then we have

$$\Delta\rho=\frac{\Delta e}{c^2}=\frac{9\mu^2\rho^2}{8\pi c^2R^6(t_0)}\simeq 10^{-127}\,\mathrm{g\,cm^{-3}} \tag{4.30}$$

At present $\Delta\rho$ is so small that it is unmeasurable and we have

$$e_{\mathrm{tot}}(t_0)\simeq e(t_0)$$

Nevertheless for radii close to minimum Δe grows rapidly and at some point the division into e and M^2 becomes meaningless because e may become negative. Knowing that

$$e_{\mathrm{tot}}=\frac{3Mc^2}{4\pi R^3(t)} \tag{4.31}$$

we calculate

$$\overset{\mathrm{tot}}{\rho}_{\max}=\overset{\mathrm{tot}}{\rho}(0)=\frac{3M}{4\pi R^3(0)}\simeq 10^{19}\,\mathrm{g\,cm^{-3}} \tag{4.32}$$

The model presented here is nonrealistic because of neglecting the electromagnetic field which plays a basic role in the Klein–Kaluza theory with torsion. A more realistic model would be a model of axial symmetry with magnetic field. By using the term (4.9) we could obtain a coupling between spin and magnetic field which is lacking in Einstein–Cartan theory. It seems that the estimations of both minimal radius and maximal density remain basically unchanged.

5. DIRAC'S EQUATION ON MANIFOLD *P*

In this chapter we deal with generalization of Dirac's equation on manifold P (metricized electromagnetic bundle). The results obtained here are similar to Thirring's results (1972). All differences will be pointed out. We will apply the generalized Dirac's Lagrangian as a source in Klein–Kaluza's theory with torsion. And then we point out that in such a way the electric dipole moment is a source of torsion in the fifth dimension. We introduce several kinds of derivatives and by using them we get a generalization of Dirac's equation. Let us start from the gauge transformation of spinors on E. We have

$$\begin{aligned} &A \to A + d\chi \\ &\psi \to \psi e^{iq\chi}, \qquad \bar{\psi} \to \bar{\psi} e^{-iq\chi} \end{aligned} \tag{5.1}$$

where $\chi: E \to R$ is a function of gauge change. Now we define spinor fields on P, Ψ and $\bar{\Psi}$ such that

$$\begin{aligned} e^*\Psi &= \psi^e, \qquad & e^*\bar{\Psi} &= \bar{\Psi}^e \\ f^*\Psi &= \psi^f, \qquad & f^*\bar{\Psi} &= \bar{\psi}^f \end{aligned} \tag{5.2}$$

where e and f are two sections of bundle P. Quantities ψ^e, $\bar{\psi}^e$, ψ^f, $\bar{\psi}^f$ are spinors taken in gauge e and f. We assume that a transition between gauges (sections) e and f is defined by the formula (2.4). Thus we have

$$\psi^f = \psi^e e^{iq\chi(x)}, \qquad \bar{\psi}^f = \bar{\psi}^e e^{-iq\chi(x)} \tag{5.3}$$

Fields ψ, $\bar{\psi}$ are defined on E:

$$\psi: E \to \mathbb{C}^4$$

whereas fields Ψ, $\bar{\Psi}$ are defined on P:

$$\Psi: P \to \mathbb{C}^4$$

And we have

$$\begin{aligned} \Psi(pg_1) &= \sigma(g_1^{-1})\Psi(p) \\ \bar{\Psi}(pg_1) &= \bar{\Psi}(p)\sigma(g_1) \end{aligned} \tag{5.4}$$

where

$$p = (x, g) \in P, \qquad g, g_1 \in U(1)$$

Obviously we have

$$\Psi(e(x)) = \pi^*(\psi^e(x))$$
$$\bar{\Psi}(e(x)) = \pi^*(\bar{\psi}^e(x)) \tag{5.5}$$

Let us define a gauge derivative of field Ψ. It is clear that

$$d\Psi = \zeta_\mu \Psi \theta^\mu + \zeta_5 \Psi \bar{\theta}^5 \tag{5.6}$$

and

$$d_1\Psi = \operatorname{hor} d\Psi = \zeta_\mu \Psi \bar{\theta}^\mu \tag{5.7}$$

where for ζ_μ we have

$$[\zeta_\mu, \zeta_\nu] = -\lambda \pi^*(F_{\mu\nu})\zeta_5 + C^\beta{}_{\mu\nu}\zeta_\beta$$
$$[\zeta_\mu, \zeta_5] = 0 \tag{5.8}$$

$C^\beta{}_{\mu\nu}$ are coefficients of nonholonomity (in Section 2 we have had only a partially nonholonomical coordinate system).

Let $\gamma_\mu \in \mathcal{L}(\mathbb{C}^4)$ be Dirac's matrices obeying the conventional relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{5.9}$$

[where $\eta_{\mu\nu}$ is Minkowski's tensor of signature $(- - - +)$] and let $B = B^+$ be a matrix such as

$$\gamma^{\mu+} = B\gamma^\mu B^{-1}, \qquad \bar{\Psi} = \Psi^+ B \tag{5.10}$$

where "+" is a Hermitian conjugation. We assume the existence of a global orthonormal coordinate system $\bar{\theta}^\mu$ on E. The infinitesimal change of frame $\bar{\theta}^\mu$ yields

$$\bar{\theta}^{\mu\prime} = \bar{\theta}^\mu + \delta\bar{\theta}^\mu = \bar{\theta}^\mu - \varepsilon^\mu_\nu \bar{\theta}^\mu$$
$$\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0 \tag{5.11}$$

If spinor field ψ corresponds to coordinate system $\bar{\theta}^\mu$, and ψ' to $\bar{\theta}'^\mu$, then we have

$$\psi' = \psi + \delta\psi = \psi - \varepsilon^{\mu\nu}\sigma_{\mu\nu}\psi$$
$$\bar{\psi}' = \bar{\psi} + \delta\bar{\psi} = \bar{\psi} + \bar{\psi}\sigma_{\mu\nu}\varepsilon^{\mu\nu} \tag{5.12}$$

where

$$\sigma_{\mu\nu}=\tfrac{1}{8}[\gamma_\mu,\gamma_\nu] \tag{5.13}$$

Fields Ψ and $\bar{\Psi}$ are defined on P and P is assumed to have an orthonormal coordinate system θ^A. This coordinate system is nonholonomical; a metric tensor $\bar{g}_{AB}$ has a signature $(---+-)$ and is diagonal. Its components are equal to 0 ± 1.

It is easy to see that

$$\{\gamma_A,\gamma_B\}=2\bar{g}_{AB} \tag{5.14}$$

where $\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4\in\mathfrak{L}(\mathbb{C}^4)$ and $\gamma^A=(\gamma^\alpha,\gamma^5)$; for γ^5 we also have

$$\gamma^{5+}=B\gamma^5B^{-1} \quad \text{and} \quad \bar{\Psi}=\Psi^+B \tag{5.15}$$

So

$$\gamma^{A+}=B\gamma^AB^{-1} \tag{5.16}$$

We perform an infinitesimal change of frame Θ^A.

$$\theta^{A'}=\theta^A+\delta\theta^A=\theta^A-\varepsilon^A_B\theta^B \qquad \varepsilon_{AB}+\varepsilon_{BA}=0 \tag{5.17}$$

If spinor field Ψ corresponds to θ^A, and Ψ' to $\theta^{A'}$, then we have

$$\Psi'=\Psi+\delta\Psi=\Psi-\varepsilon^{AB}\hat{\sigma}_{AB}\Psi$$

$$\bar{\Psi}'=\bar{\Psi}+\delta\bar{\Psi}=\bar{\Psi}+\bar{\Psi}\hat{\sigma}_{AB}\varepsilon^{AB} \tag{5.18}$$

where

$$\hat{\sigma}_{AB}=\tfrac{1}{8}[\gamma_A,\gamma_B]$$

Notice that the dimension of spinor space for $2n$-dimensional space is 2^n and it is the same for a $(2n+1)$-dimensional one. We take a spinor field for a five-dimensional space P and assume that dependence on the fifth dimension is trivial, i.e., (5.4). Taking a section we obtain spinor fields on E (the same dimension of spinor space). Spinor fields Ψ, $\bar{\Psi}$ transform on P according to formula (5.18) and spinor fields ψ, $\bar{\psi}$ on E according to (5.12). In the case of any gauge group G the situation becomes more complicated—after projecting on E we obtain several different spinor fields. We shall discuss this in Section 6.

Now let us take differentials of fields ψ and $\bar{\psi}$ with respect to $\tilde{\bar{\omega}}\alpha\beta$ (Riemannian on E). We have

$$\tilde{\bar{D}}\psi = d\psi + \tilde{\bar{\omega}}_{\alpha\beta}\sigma^{\alpha\beta}\psi$$

$$\tilde{\bar{D}}\bar{\psi} = d\bar{\psi} - \tilde{\bar{\omega}}_{\alpha\beta}\bar{\psi}\sigma_{\alpha\beta} \tag{5.19}$$

and introduce a Lagrange form of Dirac's field only

$$\mathfrak{L}_D(\psi, \bar{\psi}, d) = i\frac{\hbar c}{2}(\bar{\psi}l\wedge\, d\psi + d\bar{\psi}\wedge l\psi) + m\bar{\eta}\bar{\psi}\psi \tag{5.20}$$

where $l = \gamma_\mu\bar{\eta}^\mu$. A coupling between ψ and gravitation is introduced by replacing a normal exterior differential by a covariant exterior differential $\tilde{\bar{D}}$ (i.e., Riemannian in general relativity theory) or $\bar{D}$ (in Einstein–Cartan theory) in $\mathfrak{L}_D$. Consider a covariant derivation of spinor fields Ψ and $\bar{\Psi}$ on P. We have

$$\tilde{D}\Psi = d\Psi + \tilde{\omega}^{AB}\hat{\sigma}_{AB}\Psi$$

$$\tilde{D}\bar{\Psi} = d\bar{\Psi} - \tilde{\omega}^{AB}\bar{\Psi}\hat{\sigma}_{AB} \tag{5.21}$$

with respect to Riemannian connection $\tilde{\omega}_{AB}$. An analogous formula holds with respect to connection ω_{AB}, i.e., the Cartanian one on P. Now introduce derivatives $\mathfrak{D}$ (as in Section 3 for W_A), i.e., "gauge" derivatives:

$$\tilde{\mathfrak{D}}\Psi = \operatorname{hor}\tilde{D}\Psi$$

$$\tilde{\mathfrak{D}}\bar{\Psi} = \operatorname{hor}\tilde{D}\bar{\Psi} \tag{5.22}$$

Using (2.16) we obtain

$$\tilde{\mathfrak{D}}\Psi = \tilde{\bar{\mathfrak{D}}}\Psi - \tfrac{1}{8}\lambda F^\alpha{}_\mu[\gamma_\alpha, \gamma_5]\Psi\bar{\theta}^\mu$$

$$\tilde{\mathfrak{D}}\bar{\Psi} = \tilde{\bar{\mathfrak{D}}}\bar{\Psi} + \tfrac{1}{8}\lambda F^\alpha{}_\mu\bar{\Psi}[\gamma_\alpha, \gamma_5]\bar{\theta}^\mu \tag{5.23}$$

The derivative $\tilde{\bar{\mathfrak{D}}}$ is a covariant derivative with respect to both $\pi^*(\tilde{\bar{\omega}}_{\alpha\beta})$ and "gauge" at once. It introduces an interaction between electromagnetic and gravitational fields with Dirac's spinor in a classical already known way.

Now let us turn to the Lagrange form (5.20) and lift it on manifold P. In order to do this we have to pass from d to d_1 and from spinors $\psi, \bar{\psi}$ to $\Psi, \bar{\Psi}$. Then we obtain the Dirac Lagrangian with an electromagnetic coupling in a classical form. Nevertheless we still can work with the derivative $\tilde{\mathfrak{D}}$ defined by the formula (5.22). In such a case a Dirac Lagrangian takes the form

$$\mathfrak{L}_{\tilde{\mathfrak{D}}}(\Psi,\overline{\Psi},\tilde{\mathfrak{D}})=\tfrac{1}{2}i\hbar c\left(\overline{\Psi}l\wedge\tilde{\mathfrak{D}}\Psi+\tilde{\mathfrak{D}}\overline{\Psi}\wedge l\Psi\right)+m\Psi\overline{\Psi}\eta \qquad (5.24)$$

where $l=\eta_\mu\gamma^\mu=\bar{\eta}_\mu\gamma^\mu\wedge\theta^5$. Obviously we may also take $\tilde{\overline{\mathfrak{D}}}$ instead of $\tilde{\mathfrak{D}}$. However, this will not lead to new effects. Using formulas (5.23) we obtain after some algebra

$$\mathfrak{L}_D(\Psi,\overline{\Psi},\tilde{\mathfrak{D}})=\mathfrak{L}_D(\Psi,\overline{\Psi},\tilde{\overline{\mathfrak{D}}})-i\frac{2(G\hbar)^{1/2}}{c}F^{\mu\nu}\overline{\Psi}\gamma_5\sigma_{\mu\nu}\Psi\eta \qquad (5.25)$$

The Lagrangian $\mathfrak{L}_D(\Psi,\overline{\Psi},\tilde{\mathfrak{D}})$ describes the interaction between both electromagnetic and gravitational fields with spinor fields. $\mathfrak{L}_D(\Psi,\overline{\Psi},\tilde{\overline{\mathfrak{D}}})$ is of course of well-known classical form.

It is clear now that by working with $\tilde{\overline{\mathfrak{D}}}$ we get no additional term in the Lagrangian, namely,

$$-i\frac{2G^{1/2}}{c}\hbar F^{\mu\nu}\overline{\Psi}\gamma_5\sigma_{\mu\nu}\Psi \qquad (5.26)$$

It is an interaction of an electromagnetic field with a dipole electric moment:

$$-\frac{2G^{1/2}}{c}\hbar=-2\frac{l_{pl}}{\alpha^{1/2}}q\simeq-10^{-32}q\,[\mathrm{cm}]$$

where q is the elementary charge. So we see that using spinors Ψ and $\overline{\Psi}$ and a derivative $\tilde{\mathfrak{D}}$ in Klein–Kaluza theory, we have achieved an additional gravitational-electromagnetic effect. It is just the existence of the electric moment of a fermion, which value is composed of elementary constants (only!). Thirring (1972) also has achieved in his paper a dipole electric moment of a fermion of the same order. In his theory a minimal rest mass of a fermion is so big that it can be measured in micrograms. The Thirring dipole electric moment has a reverse sign compared to that given by our theory. Apart from a dipole electric moment Thirring has also obtained the anomalous magnetic moment of similar order. In Thirring's theory both anomalous momenta depend on the rest mass of a fermion. And to obtain a dipole electric moment of the order 10^{-32} the rest mass of a fermion usually has to be about 1 μg (minimal Thirring rest mass). In some other cases (larger rest mass) moments may be even smaller. Notice that in our case, the mass m may be arbitrary, e.g., $m=0$, and the value of our dipole electric moment depends only on elementary constants. It is also worth noticing that Thirring's quantities Ψ and $\overline{\Psi}$ have nothing to do with our spinor fields Ψ and $\overline{\Psi}$ because of the mysterious quantity φ which is absent in our theory. This quantity φ appeared in Thirring's definition of the parity operator.

Now let us consider operations of reflections defined on a manifold P. To carry this out we choose a local coordinate system on P

$$x^A=(x^\alpha, x^5), \qquad x^\alpha=(\mathbf{x}, t)$$

Then

$$\Psi(p)=\Psi(x^A)=\Psi\big((\mathbf{x}, t), x^5\big) \tag{5.27}$$

and define the transformations space reflection Π, time reversal T, charge reflection C, and combined transformations $\Pi C, \Theta=TC\Pi$ in the following way:

$$\Psi^C(x^\alpha, x^5)=C\Psi^*(x^\alpha, -x^5) \tag{5.28}$$

where $C^{-1}\gamma^\mu C=-\gamma^{\mu *}$. Taking any section f we get

$$(\psi^f)^C(x^\alpha)=C\psi^{f*}(x^\alpha)$$

and a charge changes a sign. The reflection in coordinate x^5 as a charge reflection has already been suggested by J. Rayski (1965). For the space coordinate reflection we have

$$\Psi^\Pi(x^\alpha, x^5)=\gamma^4\Psi(-\mathbf{x}, t, x^5) \tag{5.29}$$

Taking section f we obtain

$$(\psi^f)^\Pi(\mathbf{x}, t)=\gamma^4\psi^f(-\mathbf{x}, t)$$

i.e., a normal operator of parity on E.

Thirring was forced to change the definition of a parity operator on five-dimensional space, and he could not obtain a normal parity operator on E. For the transformation of time reversal T we have

$$\Psi^T(\mathbf{x}, t, x^5)=C^{-1}\gamma^1\gamma^2\gamma^3\Psi^*(\mathbf{x}, -t, -x^5) \tag{5.30}$$

Taking section f we get

$$(\psi^f)^T(\mathbf{x},t)=C^{-1}\gamma^1\gamma^2\gamma^3(\psi^f)^*(\mathbf{x},-t)$$

and a charge changes a sign, e.g., a normal time-reversal operator on space-time. For the transformation $\Theta=\Pi CT$ we put

$$\psi^\Theta(\mathbf{x},t,x^5)=-i\gamma^5\Psi(-\mathbf{x},t,x^5) \tag{5.31}$$

Taking section f we obtain

$$(\psi^f)^\Theta(\mathbf{x},t)=-i\gamma^5\psi^f(-\mathbf{x},t)$$

In both cases T and Θ Thirring could not get normal transformations T and Θ on space-time, because of introducing a quantity φ. For the transformation ΠC we get

$$\Psi^{\Pi C}(\mathbf{x},t,x^5)=\gamma^4C\Psi^*(-\mathbf{x},t,-x^5) \tag{5.32}$$

Taking a section we have

$$(\psi^f)^{\Pi C}(\mathbf{x},t)=\gamma^4C(\psi^f)^*(-\mathbf{x},t)$$

and a charge changes a sign. It is clear that the transformations obtained by us do not differ from those known from the literature. The additional term in Lagrangian (5.25) breaks symmetry ΠC or T in an analogous way as in Thirring's (1972) theory, but Thirring defines operator ΠC in a different way. This can be easily seen by acting on both sides of (5.25) with operator ΠC defined by (5.32). Of course this breaking is very weak and it cannot be linked to nonconservation of ΠC in the decays of mesons K. Nevertheless nonconservation of ΠC in these decays has good support in six-quark models: the appearance of this dipole electrical moment should rather be related to quite different, more basic gravitational-electromagnetic effects than to weak interactions of hadrons. At present the dipole electrical moment of neutrons (indirectly of quarks) is being sought in experiments. The most recent tests reveal that the dipole electrical moment of neutron is smaller than 3×10^{-24} [cm] q.

Now we apply the Klein–Kaluza theory from Section 3 to Lagrangian (5.24) or (5.25). Define the gauge differentials

$$\mathcal{D}\Psi=\operatorname{hor}D\Psi$$
$$\mathcal{D}\overline{\Psi}=\operatorname{hor}D\overline{\Psi} \tag{5.33}$$

where D is a covariant exterior differential with respect to ω_{AB} with torsion. Substituting formula (3.22) into (5.33) we get

$$\mathfrak{D}\overline{\Psi} = \overline{\mathfrak{D}}\overline{\Psi} + \tfrac{1}{8}\lambda H^{\alpha}{}_{\mu}\overline{\Psi}[\gamma_{\alpha}, \gamma_5]\theta^{\mu}$$

$$\mathfrak{D}\Psi = \overline{\mathfrak{D}}\Psi - \tfrac{1}{8}\lambda H^{\alpha}{}_{\mu}[\gamma_{\alpha}, \gamma_5]\Psi\theta^{\mu} \tag{5.34}$$

where

$$\overline{\mathfrak{D}}\Psi = \text{hor}\,\overline{D}\Psi \tag{5.35}$$

By replacing $\tilde{\mathfrak{D}}$ by $\mathfrak{D}$ we get from (5.24)

$$\mathfrak{L}_D(\Psi, \overline{\Psi}, \mathfrak{D}) = \tfrac{1}{2}i\hbar c(\overline{\Psi}l \wedge \mathfrak{D}\Psi + \mathfrak{D}\overline{\Psi} \wedge l\Psi) + m\overline{\Psi}\Psi\eta \tag{5.36}$$

Using formulas (5.34) we obtain

$$\mathfrak{L}_D(\Psi, \overline{\Psi}, \mathfrak{D}) = \mathfrak{L}_D(\Psi, \overline{\Psi}, \overline{\mathfrak{D}}) - i\frac{2G^{1/2}}{c}\hbar F_{\mu\nu}\overline{\Psi}\gamma_5\sigma^{\mu\nu}\Psi - 2i\hbar CK_{\mu\nu}\overline{\Psi}\gamma_5\sigma^{\mu\nu}\Psi \tag{5.37}$$

where $\mathfrak{L}_D(\Psi, \overline{\Psi}, \overline{\mathfrak{D}})$ describes the interaction of Dirac's spinor field Ψ with both electromagnetic and gravitational fields in Einstein–Cartan theory. It is worth noticing that in (5.37) there has appeared a term which couples a dipole electrical moment to a torsion related to the fifth dimension.

Applying the theory from Section 3 we write down (3.45), i.e., Cartan's equation for sources given by (5.36). In effect we have

$$\overline{Q}^{\gamma}_{\alpha\beta} = i\frac{8\pi G}{c^3}\hbar^3\overline{\Psi}(\gamma^{\gamma}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma^{\gamma})\Psi$$

$$= -i\frac{4\pi G}{c^3}\hbar\overline{\Psi}\bar{\eta}_{\gamma\alpha\beta\mu}\gamma^5\gamma^{\mu}\Psi \tag{5.38}$$

and

$$K^{\alpha\beta} = -\frac{4\pi G^{1/2}}{c^2}M^{\alpha\beta} = -\frac{8\pi G}{c^3}\hbar i\overline{\Psi}\gamma_5\sigma^{\alpha\beta}\Psi \tag{5.39}$$

The equation of "matter" $L^A=0$ in this case yields [using (5.39) and (5.38)]

$$i\hbar c\gamma^\mu\left(\tilde{\overline{\nabla}}_\mu - iqA_\mu\right)\Psi + i\frac{2G^{1/2}}{c}\hbar F^{\mu\nu}\gamma^5\sigma_{\mu\nu}\Psi + \\ +4\pi l_{\text{Pl}}^2 c\left[\left(\overline{\Psi}\gamma^5\gamma^\gamma\Psi\right)\gamma^\mu \overset{*}{\sigma}_{\mu\gamma}\Psi + 4\left(\overline{\Psi}\gamma^5\sigma^{\mu\nu}\Psi\right)\gamma^5\sigma_{\mu\nu}\psi\right] + m\Psi = 0 \tag{5.40}$$

where $l_{\text{Pl}} = G^{1/2}\hbar^{1/2}c^{-3/2} \simeq 10^{-33}$ [cm] is a Planck length and

$$\overset{*}{\sigma}_{\nu\mu} = \tfrac{1}{2}\overline{\eta}^{\,\nu\mu}{}_{\alpha\beta}\sigma^{\alpha\beta} \tag{5.41}$$

is a dual tensor. Notice that by putting $m=0$ into (5.40) we obtain an equation that looks like a nonlinear equation of Heisenberg's prematter theory (Urmaterie-gleichungen), with a nonlinear term resulting from the Einstein–Cartan theory and Klein–Kaluza theory with torsion. The second pair of Maxwell's equations takes the form

$$\tilde{\overline{\nabla}}_\mu H^{\nu\mu} = \frac{4\pi}{c}\overset{\text{tot}}{j^\nu} \tag{5.42}$$

where

$$\overset{\text{tot}}{j^\nu} = cq\overline{\Psi}\Gamma^\nu\Psi$$

and

$$\Gamma^\nu = \gamma^\nu - l_{\text{Pl}}^2\left[\frac{8i}{q}\overset{*}{F}{}^{\nu\mu} - \frac{\pi l_{\text{Pl}}}{\alpha^{1/2}}\left(\overline{\Psi}\gamma^5\overset{*}{\sigma}{}^{\nu\mu}\Psi\right)\right]\gamma_\mu\gamma^5$$

and

$$H^{\nu\mu} = F^{\nu\mu} - i\frac{8\pi G^{1/2}}{c}\hbar\overline{\psi}\gamma^5\sigma^{\nu\mu}\psi$$

and

$$\alpha = \frac{q^2}{\hbar c} \simeq \frac{1}{137}$$

is a fine-structure constant and

$$\overset{*}{F}{}^{\nu\mu} = \tfrac{1}{2}\bar{\eta}^{\nu\mu}{}_{\alpha\beta}F^{\alpha\beta}$$

is a dual tensor. We may regard Γ^{ν} as a specific "vertex function" introducing a certain structure of electrical charge which is originated from Klein–Kaluza theory with torsion. Notice also that Γ^{ν} we deal with has a component breaking parity conservation

$$\frac{\pi l_{\mathrm{Pl}}^{3}}{\alpha^{1/2}}\left(\bar{\Psi}\gamma^{5}\overset{*}{\sigma}{}^{\nu\mu}\Psi\right)\gamma_{\mu}\gamma^{5} \tag{5.43}$$

Since the Planck's length l_{Pl} appears as l_{Pl}^{3} in (5.43) then parity is breaking very weakly indeed. Finally we notice that the introducing of derivatives $\tilde{\mathfrak{D}}$ and $\mathfrak{D}$ can be regarded as a generalization of minimal coupling. Because of this generalization we obtained several new effects.

6. GENERALIZATION TO ANY GAUGE GROUP

In this chapter we generalize certain results obtained in Sections 2, 3, and 5 by considering an arbitrary gauge group G instead of $U(1)$.

Let us consider a principal fiber bundle P over E with a structural group G, metricized as in the Trautman (1970) work. (See also Section 1.) Next we introduce a linear Riemannian connection on P. We build a $(n+4)$ form of curvature scalar for this connection and vary it with respect to metric tensor, a frame, and a connection. As in Section 2, we define a nonholonomical natural frame:

$$\theta^{A} = \left(\pi^{*}(\bar{\theta}^{\alpha}), \lambda\theta^{a}\right), \qquad \alpha = 1,2,3,4, \qquad a = 5,6,\ldots,n+4 \tag{6.1}$$

where $n = \dim G$.

$\omega = \theta^{a}X_{a}$ is a connection of principal bundle P and X_{a} are generators of Lie algebra of G in an adjoined representation. Similarly as in the works by Cho and Jang (1975), Cho and Freund (1975), Cho (1975), and Kerner (1968), we obtain equations similar to those of five-dimensional theory.

The right-hand side of the gravitation equation contains a tensor of energy-momentum of gauge field and a cosmological term. This term vanishes when the group G is Abelian. The cosmological constant in the Einstein equations obtained in such a theory is very large ($\sim 10^{33}$), which weakens these results.

Probably, the geometrization of spontaneous symmetry breaking, of Higgs' field and the "Higgs" mechanism, could decrease this too-large constant. In this theory we obtain also equations of the Yang–Mills fields (an analog of the second pair of Maxwell's equations). Derivatives in these equations are both with respect to the Riemannian connection of E and "gauge." In the case of $U(1)$ these derivatives become the usual derivatives with respect to the Riemannian connection on E. Observe also that in the case of any gauge group G the strength of the gauge field (curvature of a connection on bundle P) is defined on P, and it is a form with values in Lie's algebra of G. This form evidently depends on a choice of section, and that is why it is more convenient to define all quantities on P rather than on E. In the case with nonvanishing torsion we introduce a non-Riemannian, but metrical, connection on P. We also assume the horizontality of a two-form of torsion.

$$\Theta^A = \operatorname{hor} \Theta^A \tag{6.2}$$

Next we introduce a certain horizontal two-form with values in Lie's algebra of G:

$$\Theta = \Theta^a X_a \tag{6.3}$$

Thus the torsion separates into two independent parts

$$\pi^*(\overline{\Theta}^\alpha) \text{ and } \Theta$$

where $\overline{\Theta}^\alpha$ is a torsion of space-time and Θ describes torsion in higher dimensions. As far as Θ is concerned we also assume that it is of ad type like ω and Ω.

In this way a linear metrical connection ω_{AB} is defined on P, whose torsion has been given in terms of $\overline{\Theta}^\alpha$ and Θ. This connection is a generalization of the connection discussed in Section 3 to the case of any gauge group G.

When $G = U(1)$ then $\Theta = \Theta^5$ and the condition ad for Θ becomes (1.2) for form Θ^5. As in the electromagnetic case we introduce a two-form with values in Lie's algebra of G that is an analog of the second two-form of the strength of the electromagnetic field:

$$H = \Omega - \tfrac{1}{2}\Theta \tag{6.4}$$

By generalizing the results of Section 3 we build K, a $(n+4)$-form of curvature scalar and introduce Λ, a $(n+4)$-form of sources (Lagrangian).

By varying

$$\int_{\mathrm{VCP}}(K-8\pi\Lambda)$$

with respect to metric, connection and frame we obtain some field equations which differ from the equations of Section 3 in the following aspects. On the right-hand side of the equations of a gravitational field, instead of Israel's tensor, there is a tensor of energy-momentum of gauge field with polarization

$$\bar{t}^{\alpha}=\frac{1}{4\Pi}\left(h_{ab}H^{a\mu\beta}F^{b\alpha}{}_{\mu}\bar{\eta}_{\beta}-\tfrac{1}{4}h_{ab}F^{a}_{\mu\beta}F^{b\mu\beta}\bar{\eta}\alpha\right) \tag{6.5}$$

where

$$e^{*}\Omega=\tfrac{1}{2}\left(F^{a}_{\mu\nu}\bar{\theta}^{\mu}\wedge\bar{\theta}^{\nu}\right)X_{a}$$

$$e^{*}H=\tfrac{1}{2}\left(H^{a}_{\mu\nu}\bar{\theta}^{\mu}\wedge\bar{\theta}^{\nu}\right)X_{a}$$

and e is a section of P. There is of course an additional term that is related to higher dimensions and is a square of torsion in higher dimensions

$$\Pi g_{\alpha\beta}h_{ab}M^{a}_{\mu\nu}M^{b\mu\nu} \tag{6.6}$$

where $M^{a}_{\mu\nu}$ is associated with torsion in higher dimensions:

$$K^{a}_{\mu\nu}=-4\pi M^{a}_{\mu\nu} \tag{6.7}$$

where

$$\tfrac{1}{2}\left(K^{a}_{\mu\nu}\bar{\theta}^{\nu}\wedge\bar{\theta}^{\nu}\right)X_{a}=e^{*}\Theta$$

Torsion associated with higher dimensions has as a source a polarization of the gauge field. As far as the equations of the gauge field (the second pair) are concerned, the derivatives which appear in the equations are taken with respect to both "gauge" and the metrical connection on E lifted on P. This means that there will be an additional internal current obeying the analog of "Ohm's law" for a gauge field:

$$\frac{gauge}{\nabla_{\mu}}H^{a\mu}_{\nu}=4\pi j^{a}_{\nu} \tag{6.8}$$

This current $j^{a\nu}_{i}=-\frac{1}{4}S^{\nu}{}_{\alpha\beta}H^{a\alpha\beta}$ would impose a structure of color charges.

Bianchi's identities for linear connection of P, as in Section 3, give rise to conservation laws of energy-momentum, angular momentum, and "color charges." Although a mathematical scheme of the above-mentioned theory of the Klein–Kaluza type with torsion is clear and easily achieved from the formalism of Section 3, it is not known what physical sense is hidden behind the quantities $M^a_{\mu\nu}$.

We can give a generalization of the example discussed in Section 3.7; namely, we assume that the vector field W_A transforms under the group G instead of $U(1)$. This field describes a vector field W_α on manifold E and a multiplet of scalar field φ_a that all transform under the same group G. In the same way as in Section 3 we can obtain sources of torsion in higher dimensions and analogous nonlinear effects.

Let us consider $G=SU(2)\times U(1)$ and generalize results presented in Section 5. We have $\dim P=8$ now. According to J. Rayski's suggestions (1977), we analyze spinors defined on P. The dimension of spinor space is $2^4=16$. Introducing spinor fields $\Psi, \overline{\Psi}$ on P we can treat them as multiplets of fermions:

$$e^*\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix} \tag{6.9}$$

where e is a section of P. Introducing Clifford's algebra Γ^A, $A=1,2,3,\ldots,8$ for a form invariant under transformations of $SO(1,7)$

$$\{\Gamma^A,\Gamma^B\}=2\eta^{AB} \qquad \Big(-\underbrace{+++++++}_{\times 7}\Big) \tag{6.10}$$

we find generators of Lie's algebra of $SO(1,7)$

$$\sigma^{AB}=\tfrac{1}{8}[\Gamma^A,\Gamma^B] \tag{6.11}$$

As in Section 5, we assume that transformations of Ψ and $\overline{\Psi}$ correspond to the transformation of a global orthonormal frame on P and Θ^A—formulas (5.17) and (5.18). Obviously in such a case we have $SO(1,7)$ instead of $SO(1,4)$.

Introducing differentials $\mathfrak{D}$, as in Section 5, we obtain new terms in the Dirac Lagrangian for spinors Ψ and $\overline{\Psi}$. On a space-time they will separate according to formulas (6.9); this will give rise to new interactions between fermions and gauge field.

Simultaneously we can associate the torsion in higher dimensions with certain terms built from those spinor fields. It seems that such an approach offers, in principle, at least some hope for building a unified theory of gravitational, weak, and electromagnetic interactions. Unfortunately, we shall not expect success in this direction until both spontaneous symmetry breaking and Higgs' mechanism are geometrized. All the fields discussed in the paper–i.e., both boson and fermion fields—are massless, and it is only Higgs' mechanism that can give them masses without breaking gauge symmetry.

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